The p-adic zeta function of Kubota-Leopoldt and more functions

In these notes we are going to extend the Riemann ζ function $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ to p-adic numbers using a tecnique of p-adic interpolation. Then we will describe some Banach spaces of function over $\mathbb{Z}_p$ that correspond to the usual "real" Banach spaces $C^k(\mathbb{R})$

1 First naive tentatives

The first naive tentative to define a p-adic Riemann function would be to take, for all positive n, the function $f(s) = n^s$ and extend it to p-adic integers and then take the sum of the inverses of these functions. Anyway this approach does not work. Infact we have

Lemma: Let $\{s_i\}$ be a strictly increasing sequence of positive integers converging to $s \in \mathbb{Z}_p$. Then the sequence p^{s_i} converges to 0 with respect to the p-adic norm.

Therefore we should take $p^s = 0 \ \forall s \in \mathbb{Z}_p, \ s \notin \mathbb{Z}_p$, that is clearly absurd. Therefore the function n^s cannot be p-adically interpolated if $p|n$. Anyway it is easy to show that n^s extends to a continous function over $\mathbb{Z}_p$ if p does not divide n.

We can try to separate the multiple-to-p terms from the other ones in the sum, obtaining

$$\zeta(s) = \sum_{n \notin p\mathbb{Z}} \frac{1}{n^s} + \sum_{m=1}^{+\infty} \frac{1}{pm^s} = \sum_{n \notin p\mathbb{Z}} \frac{1}{n^s} + \frac{1}{p^s}\zeta(s), \tag{1}$$

and therefore having

$$\zeta(s) = \frac{1}{1-p^{-s}} \sum_{n \notin p\mathbb{Z}} \frac{1}{n^s}, \tag{2}$$

that is, we have taken out the p-Euler factor in the definition of ζ. Anyway it can be shown that the sum in (2) diverges in $\mathbb{Z}_p$, so even this definition does not work.

We can try another approach: we know that $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \in \mathbb{Q}$ for all positive integers n. So we can try to interpolate the right hand side of this formula and then use the density of $\mathbb{N}$ in $\mathbb{Z}_p$ to obtain a continuous function. (This is the approach used in Koblitz's book). Anyway this approach requires the theory of p-adic distributions, so we are not going to use it. But we will use a similar idea, that is, express the quantities $\zeta(-n)$ as a formula that we can easily interpolate to a continous p-adic function.

2 The Amice transform

In this section we use the Amice transform $\mu \to \int_{\mathbb{Z}_p} (1+T)^x \mu(x)$ to build up a measure on p-adic integers.

Lemma: $\forall a \in \mathbb{Z}_p^*$ there exists a measure $\lambda_a \in D^0(\mathbb{Z}_p, \mathbb{Q}_p)$ such that $A_{\lambda_a} = \frac{1}{T} - \frac{a}{(1+T)^a - 1}$.

Proof: Denote as $F(T)$ the function on the right. Since the Amice transform is an isometry between the set of measures on $\mathbb{Z}_p$ and the set of power series with bounded coefficients in $\mathbb{Q}_p$, it is sufficient to show that $F(T) \in \mathbb{Z}_p[[T]]$. We have that

$$F(T) = \frac{1}{T} - \frac{a}{\sum_{n=1}^{+\infty} \binom{a}{n} T^n} = \frac{1}{T} - \frac{1}{T \sum_{n=1}^{+\infty} a^{-1} \binom{a}{n} T^{n-1}} = \frac{\sum_{n=2}^{+\infty} a^{-1} \binom{a}{n} T^{n-2}}{\sum_{n=1}^{+\infty} a^{-1} \binom{a}{n} T^{n-1}}. \quad (3)$$

This series belongs to $\mathbb{Z}_p[[T]]$ because the denominator has constant term equal to 1 and therefore lies in $\mathbb{Z}_p[[T]]^*$. So the measure λ_a exists. Moreover, using again the fact that the Amice transform is an isometry, we have that $v_{D^0}(\lambda_a) = 0$.

Proposition: $\forall a \in \mathbb{Z}_p^*, n \in \mathbb{N}$ we have

$$\int_{\mathbb{Z}_p} x^n \lambda_a = (-1)^n (1 - a^{1+n}) \zeta(-n). \quad (4)$$

Proof: Let $a \in \mathbb{R}^+$. We define the auxiliary real function $f_a(t) = A_{\lambda_a}(e^t - 1) = \frac{1}{e^t - 1} - \frac{a}{e^{at} - 1}$. Then $t^n f_a(t) \to 0$ for every positive n.

It follows that the analytic complex L-function

$$L(f_a, s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_a(t) t^s \frac{dt}{t} = (1 - a^{1-s})\zeta(s) \tag{5}$$

satisfies

$$f_a^{(n)}(0) = (-1)^n L(f_a, -n) = (-1)^n (1 - a^{1+n})\zeta(-n). \tag{6}$$

The last expression is an algebraic identity, so it holds even when we take $a \in \mathbb{Z}_p^*$. Therefore finally we have

$$f_a^{(n)}(0) = \frac{d}{dt}^{(n)} A_{\lambda_a}(e^t - 1)|_{t=0} = \frac{d}{dt}^{(n)} (\int_{\mathbb{Z}_p} e^{tx} \lambda_a)|_{t=0} = \int_{\mathbb{Z}_p} x^n \lambda_a. \tag{7}$$

The last equality proves the proposition.

As a corollary of this result we can prove the famous Kummer congruences.

Corollary(Kummer congruences): $\forall a \in \mathbb{Z}_p^*, k \geq 1$ (or $k \geq 2$ if $p = 2$), let $n_1, n_2 \geq k$ such that $n_1 \equiv n_2 \pmod{p^{k-1}(p-1)}$. Then

$$v_p((1 + a^{1+n_1})\zeta(-n_1) - (1 - a^{1+n_2})\zeta(-n_2)) \geq k. \tag{8}$$

Proof: Apply the proposition to express the left hand side of the formula in the integral form. Take the p-adic valuation and note that n_1 and n_2 must have the same parity, therefore the terms $(-1)^{n_i}$ have the same sign and disappear when taking the valuation. Then the formula becomes

$$v_p(\int_{\mathbb{Z}_p} (x^{n_1} - x^{n_2})\lambda_a) \geq v_{D^0}(\lambda_a) + v_{C^0}(x^{n_1} - x^{n_2}) = \inf_{x \in \mathbb{Z}_p} v_p(x^{n_1} - x^{n_2}) \tag{9}$$

therefore we only need to show that the last infimum is $\geq k$. We distinguish two cases:

1) If $x \in p\mathbb{Z}_p$ then p^k divides both x^{n_1} and x^{n_2} and we have the inequality.

2) If $x \in \mathbb{Z}_p^*$ then $x^{p^{k-1}(p-1)} \equiv 1 \pmod{p^k}$ and therefore we have that $v_p(x^{n_1} - x^{n_2}) = v_p(1 - x^{n_2 - n_1}) \geq k$ and we are done.

3 Operations on measures

We will know list some fundamental operations on the set of measures, that we are going to apply to λ_a. In this section we always take $\mu \in D^0(\mathbb{Z}_p, \mathbb{Q}_p)$

- **Restriction:** Let $X \in \mathbb{Z}_p$ be a compact open set. The *restriction of μ to X* is defined as the composite of μ with multiplication by the characteristic function of the set X. In particular, if $X = i + p^n\mathbb{Z}_p$ for $0 \leq i \leq p^n - 1$, then

$$1_{i+p^n\mathbb{Z}_p}(x) = \frac{1}{p^n} \sum_{z^{p^n}=1} z^{x-i} \tag{10}$$

  and therfore computing the Amice transform, we have

$$A_{Res_{i+p^n\mathbb{Z}_p}(\mu)}(T) = \frac{1}{p^n} \sum_{z^{p^n}=1} \frac{1}{z^i} A_\mu((1+T)z - 1). \tag{11}$$

- *The ϕ-action:* Let $\phi : C^0(\mathbb{Z}_p, \mathbb{Q}_p) \to C^0(\mathbb{Z}_p, \mathbb{Q}_p)$ defined as $\phi(F)(T) = F((1+T)^p - 1)$. We define the action of ϕ on μ by

$$\int_{\mathbb{Z}_p} f(x)\phi\mu(x) = \int_{\mathbb{Z}_p} f(px)\mu(x) \tag{12}$$

  for all $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$. It follows that

$$A_{\phi\mu}(T) = A_\mu((1+T)^p - 1) = \phi(A_\mu(T)). \tag{13}$$

- *The ψ-action:* Let $\psi : C^0(\mathbb{Z}_p, \mathbb{Q}_p) \to C^0(\mathbb{Z}_p, \mathbb{Q}_p)$ defined as

$$\psi(F)((1+T)^p - 1) = \frac{1}{p} \sum_{z^p=1} F((1+T)z - 1). \tag{14}$$

  We define the action of ψ on μ by

$$\int_{\mathbb{Z}_p} f(x)\psi\mu(x) = \int_{\mathbb{Z}_p} f(\frac{x}{p})\mu(x) \tag{15}$$

  for all $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$. It follows that $A_{\psi\mu}(T) = \psi(A_\mu(T)$.

- *The γ-action:* Let $\Gamma = Gal(\mathbb{Q}_p(\mu_p^\infty)/\mathbb{Q}_p)$ and let $\chi : \Gamma \to \mathbb{Z}_p^*$ be the cyclotomic character. $\forall \gamma \in \Gamma$ we define the action of γ on μ by
$$\int_{\mathbb{Z}_p} f(x)\gamma\mu(x) = \int_{\mathbb{Z}_p} f(\chi(\gamma)x)\mu(x). \tag{16}$$
and we have that
$$A_{\gamma\mu}(T) = A_\mu((1+T)^{\chi(\gamma)} - 1) = \gamma(A_\mu(T)). \tag{17}$$

Properties:

1. $\psi\phi\mu = \mu$ that is $\psi \circ \phi = Id$.
2. $\phi\psi\mu = \mu - Res_{\mathbb{Z}_p^*}(\mu)$.
3. $\forall \gamma \in \Gamma$ $\phi\gamma\mu = \gamma\phi\mu$ and $\psi\gamma\mu = \gamma\psi\mu$.

4 The measure λ_a

We are now going to apply the operations defined in the previous section on the measure λ_a and see how it modifies.

Proposition: $\forall a \in \mathbb{Z}_p^*$ λ_a is a fixed point for the ψ-action, that is $\psi\lambda_a = \lambda_a$.

Proof: It suffices to show the same result for the Amice transform A_{λ_a}. Let $\gamma_a \in \Gamma$ such that $\chi(\gamma_a) = a$. We have that
$$\psi(A_{\lambda_a}(T)) = \psi(\frac{1}{T}) - \psi(\frac{a}{(1+T)^a - 1}) = \psi(\frac{1}{T}) - a\psi\gamma_a(\frac{1}{T}) \tag{18}$$
so, by property 3 of the ψ-action, we only need to show that $\psi(\frac{1}{T}) = \frac{1}{T}$. Let $F(T) = \psi(\frac{1}{T})$. We have
$$F((1+T)^p - 1) = \frac{1}{p}\sum_{z^p=1} \frac{1}{(1+T)z - 1} = \frac{-1}{p}\sum_{z^p=1}\sum_{n=0}^{+\infty}((1+T)z)^n. \tag{19}$$

Now we use the absolute convergence argument to exchange the two summands and observe that the sum of a fixed power of p-th roots of unity is null if the exponent is prime to p and equals p if p divides the exponent and we have

$$\frac{-1}{p}\sum_{n=0}^{+\infty}(1+T)^n(\sum_{z^p=1} z^n) = -\sum_{n=0}^{+\infty}(1+T)^{pn} = \frac{1}{(1+T)^p-1} \qquad (20)$$

and we are done.

Corollary:

1. $Res_{\mathbb{Z}_p^*}(\lambda_a) = (1-\phi\psi)\lambda_a = (1-\phi)\lambda_a$.
2. $\int_{\mathbb{Z}_p^*} x^n\lambda_a = \int_{\mathbb{Z}_p} x^n(1-\phi)\lambda_a = (-1)^n(1-a^{1+n})(1-p^n)\zeta(-n)$.

Proof: 1) follows immediatly from the proposition and the properties of the action of ϕ and ψ. 2) follows from 1) and the theorem of section 1.

The factor $(1-p^n)$ is no other than the p-Euler factor of the classical Riemann zeta function. Therefore, to avoid the lack of p-adic continuity of the exponential function in the integrand, we can simply reduce to $\mathbb{Z}_p^*$ *taking out* this factor.

We can now state the main result.

Main Theorem: $\forall i \in \mathbb{Z}/(p-1)\mathbb{Z}$, $i \in \mathbb{Z}/2\mathbb{Z}$ if $p=2$, there exists a unique function $\zeta_{p,i} : \mathbb{Z}_p \to \mathbb{Q}_p$ such that:

1. $\zeta_{p,i}$ is continous if $i \neq 1$;
2. $\zeta_{p,1}$ is continous up to a single pole in $s=1$;
3. $\zeta_{p,i}(-n) = (1-p^n)\zeta(-n)$ if $n \equiv -i \pmod{(p-1)}$.

5 The Leopoldt's Γ-transform

The purpose of this section is to prove the main theorem we have just stated. The idea is to extend to $\mathbb{Z}_p$ the function described in the previous corollary, but we need an argument to prove that the extension can be realized in a continuous way.

We start with a lemma that enables us to split up to exponential function into two different pieces

Lemma: Every $x \in \mathbb{Z}_p^*$ can be written uniquely in the form $\omega(x)\eta(x)$, with $\omega(x) \in \mu(\mathbb{Q}_p)$, $\eta(x) \in 1 + q\mathbb{Z}_p$, where

- $\mu(\mathbb{Q}_p) = \pm 1$ and $q = 4$ if $p = 2$;
- $\mu(\mathbb{Q}_p) = \mu_{p-1}$ and $q = p$ if $p \neq 2$.

Proof: If $p = 2$ the lemma is obvious, otherwise take $\omega(x) = \lim_{n\to+\infty} x^{p^n}$ and $\eta(x) = \omega(x)^{-1}x$.

$\omega(x)$ is called the *Teichmueller lift*, while $\eta(x) = \exp(\log(x))$. Therefore we can write $x^n = \omega(x)^n\eta(x)^n$; note that $\eta(x)^n$ can be extended to a continous p-adic function $\eta(x)^s$ with respect to s, but $\omega(x)^n$ can not be extended in p-adically continous way.

Theorem: Let i be defined as in Main Theorem , $u = 1 + q$ and $\lambda \in D^0(\mathbb{Z}_p^*, \mathbb{Q}_p^*)$. Then there exists a measure $\Gamma_\lambda^{(i)} \in D^0(\mathbb{Z}_p, \mathbb{Q}_p)$, called the i-th Leopoldt's Γ-transform of λ, such that

$$\int_{\mathbb{Z}_p^*} \omega(x)^i\eta(x)^s\lambda(x) = \int_{\mathbb{Z}_p} u^{sy}\Gamma_\lambda^{(i)}(y) = A_{\Gamma_\lambda^{(i)}}(u^s - 1). \tag{21}$$

Proof: Write $\mathbb{Z}_p^* = \bigcup_{\epsilon\in\mu(\mathbb{Q}_p)} \epsilon + q\mathbb{Z}_p$. Using the additivity of integrals and the γ-action we have

$$\int_{\mathbb{Z}_p^*} \omega(x)^i\eta(x)^s\lambda(x) = \sum_{\epsilon\in\mu(\mathbb{Q}_p)} \omega(\epsilon)^i \int_{\epsilon+q\mathbb{Z}_p} \eta(x)^s\lambda(x) = \sum_{\epsilon\in\mu(\mathbb{Q}_p)} \omega(\epsilon)^i \int_{1+q\mathbb{Z}_p} \eta(\epsilon x)^s\gamma_{\epsilon^{-1}}\lambda(x). \tag{22}$$

Observe now that there is an isomorphism $\alpha : 1 + q\mathbb{Z}_p \to \mathbb{Z}_p$ defined by $\alpha(x) = y = \frac{\log(x)}{\log(u)}$ and that $\eta(x)^s = \exp(s\log(x)) = \exp(sy\log(u)) = u^{sy} = u^{s\alpha(x)}$. Therefore we can define a new measure $\alpha_*(\gamma_{\epsilon^{-1}}\lambda)(y) = \gamma_{\epsilon^{-1}}\lambda(\alpha^{-1}(y))$ and our expression becomes

$$\sum_{\epsilon\in\mu(\mathbb{Q}_p)} \omega(\epsilon)^i \int_{1+q\mathbb{Z}_p} \eta(\epsilon x)^s\gamma_{\epsilon^{-1}}\lambda(x) = \sum_{\epsilon\in\mu(\mathbb{Q}_p)} \omega(\epsilon)^i \int_{\mathbb{Z}_p} u^{sy}\alpha_*(\gamma_{\epsilon^{-1}}\lambda)(y). \tag{23}$$

therefore we set $\Gamma_\lambda^{(i)} = \sum_{\epsilon\in\mu(\mathbb{Q}_p)} \omega(\epsilon)^i \alpha_*(\gamma_{\epsilon^{-1}}\lambda)$ and we are done.

Note that the integral in the left hand side of formula (21), which was a priori non continuous, is now known to be continuous because the right hand side is. Thanks to this result we can know prove the main theorem.

Proof of the Main Theorem: Note firstly that, if the function $\zeta_{p,i}$ exists, then it must be unique, because it is defined as a continuous function on the set $c + (p-1)\mathbb{N}$, which is dense in $\mathbb{Z}_p$. We define

$$\zeta_{p,i}(s) = \frac{1}{1-\omega(a)^{1-i}\eta(a)^{1-s}} \int_{\mathbb{Z}_p^*} \omega(x)^{-i}\eta(x)^{-s}\lambda_a(x). \tag{24}$$

If $n \equiv -i \pmod{(p-1)}$ then we have, by the corollary in the previous section

$$\zeta_{p,i}(-n) = \frac{1}{1-\omega(a)^{1+n}\eta(a)^{1+n}} \int_{\mathbb{Z}_p^*} \omega(x)^n\eta(x)^n\lambda_a(x) = (-1)^n(1-p^n)\zeta(-n). \tag{25}$$

Moreover note that $\zeta(-n) = 0$ if n is odd, so we can remove the factor $(-1)^n$ from the formula without changing the result. This gives property 3). The integral in (24) is continuous because of the previous theorem, so $\zeta_{p,i}$ is continuous unless the denominator vanishes. This happens for $i = 1,\ s = 1$ and in that case we have a simple pole. Therefore the theorem is proved.

6 C^k functions

The purpose of this chapter is to define further p-adic Banach spaces of function on $\mathbb{Z}_p$ extending the concepts of derivatives to p-adic numbers. We recall that, for $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, we can write $f(x) = \sum_{n=0}^{+\infty} a_n(f)\binom{x}{n}$, where $a_n(f) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}f(n-1)$.

We define $f^{(k)}$ the k-th derivative of f by induction as:

- $f^{(0)}(x) = f(x)$,
- $f^{(k)}(x, h_1, ..., h_k) = \frac{1}{h_k}(f^{(k-1)}(x+h_k, h_1, ..., h_{k-1}) - f^{(k-1)}(x, h_1, ..., h_{k-1})) = \frac{1}{h_1 h_2 ... h_k}(\sum_{J\subseteq\{1,...,k\}}(-1)^{|J|-1}f(x+\sum_{j\in J}h_j))$.

We say that $f \in C^k(\mathbb{Z}_p, \mathbb{Q}_p)$ if $f^{(i)}$ can be extended to a continous function on $\mathbb{Z}_p^{i+1}$ for every $i \leq k$. Observe that, for valuations, one has the inductive formula

$$v_p(f^{(k)}(x, h_1, ..., h_n)) \geq v_{C^0}(f) - \sum_{j=1}^{k} v_p(h_k). \tag{26}$$

This notion of derivative is quite different from the usual one. For example, writing an element $x \in \mathbb{Z}_p$ as $\sum_{n=0}^{+\infty} a_n(x)p^n$, consider the function $f(x) = \sum_{n=0}^{+\infty} a_n(x)p^{2n}$. Then $v_p(f(x) - f(y)) = 2v_p(x - y)$ and therefore f is C^∞ in the usual sense. But $f \notin C^2(\mathbb{Z}_p, \mathbb{Q}_p)$ according to our definition. In fact take $x_n = (0, p^n, p^n)$ and $y_n = (p^n(p-1), p^n, p^n)$; then $f^{(2)}(x_n) = 0$, $f^{(2)}(y_n) = p(1-p)$ and $v_p(x_n - y_n) \to +\infty$, but $v_p(f^{(2)}(x_n) - f^{(2)}(y_n)) \to 1 \neq +\infty = v_p(f^{(2)}(0))$. Therefore $f^{(2)}$ is not p-adically continuous.

We can define a valuation on $C^k(\mathbb{Z}_p, \mathbb{Q}_p)$ by

$$v_{C^k}(f) = \min_{i\leq k} \inf_{(x,h_1,...,h_i)\in\mathbb{Z}_p^{i+1}} v_p(f^{(i)}(x, h_1, ..., h_i)) \tag{27}$$

and define the numbers

$$L(n,k) = \max\{\sum_{j=1}^{i} v_p(n_j),\ i \leq k,\ n_j \in \mathbb{N}^+,\ \sum_{j=1}^{i} n_j = n\} \tag{28}$$

Theorem(Barski): $C^k(\mathbb{Z}_p, \mathbb{Q}_p)$ is a Banach space with the valuation v_{C^k} and a Banach basis is given by the functions $p^{L(n,k)}\binom{x}{n}$

Corollary: Let $f = \sum_{n=0}^{+\infty} a_n(f)\binom{x}{n} \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, then the follwing are equivalent:

1. $f \in C^k(\mathbb{Z}_p, \mathbb{Q}_p)$;
2. $\lim_{n\to+\infty} v_p(a_n(f)) - k\frac{\log(1+n)}{\log(p)} = +\infty$;
3. $\lim_{n\to+\infty} n^k |a_n(f)|_p = 0$.

Proof(sketch): 2 $\iff$ 3 is obvious beacuse they are the same proposition written in term of p-adic valuation and p-adic norm respectively. To show that 1 $\iff$ 2 simply note that $L(n,k)$ is asymptotically equivalent to $k\frac{\log(1+n)}{\log(p)}$.

This corollary enables us to extend the definition of p-adic derivatives even when k is not an integer. Let $f = \sum_{n=0}^{+\infty} a_n(f)\binom{x}{n} \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$ and $r \in \mathbb{R}^+$. We say that $f \in C^r(\mathbb{Z}_p, \mathbb{Q}_p)$ if

$$\lim_{n\to+\infty} n^r |a_n(f)|_p = 0 \tag{29}$$

Using the corollary again, we can define a valuation over $C^r(\mathbb{Z}_p, \mathbb{Q}_p)$ by

$$v_{C^r}(f) = \inf\{v_p(a_n(f)) - r\frac{\log(1+n)}{\log(p)}\}. \tag{30}$$

Theorem: $C^r(\mathbb{Z}_p, \mathbb{Q}_p)$ is a Banach space with respect to the valuation v_{C^r}.

We have a natural containment $C^0(\mathbb{Z}_p, \mathbb{Q}_p) \supseteq C^k(\mathbb{Z}_p, \mathbb{Q}_p) \supseteq C^r(\mathbb{Z}_p, \mathbb{Q}_p)$. if $k \leq r$.

7 Locally analytic functions

The purpose of this section is to extend the theory of Taylor power series to the p-adic complex number field $\mathbb{C}_p$. We start with a technical lemma.

Lemma: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}_p$ such that $v_p(a_n) \to +\infty$ and let $f(T) = \sum_{n=0}^{+\infty} a_n T^n$. Then the following are true:

1. If $x_0 \in O_{\mathbb{C}_p}$ then $f^{(k)}(x_0)$ converges for all k and
$$\lim_{k\to+\infty} v_p(\frac{f^{(k)}(x_0)}{k!}) = +\infty; \tag{31}$$
2. if $x, x_0 \in \mathbb{C}_p$ then
$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{32}$$
and
$$\inf\ v_p(\frac{f^{(n)}(x_0)}{n!}) = \inf\ v_p(a_n). \tag{33}$$
3. $\inf\ v_p(a_n) = \inf_{x\in O_{\mathbb{C}_p}} v_p(f(x))$ and $v_p(f(x)) = \inf\ v_p(a_n)$ for all but finitely many cosets $x \in O_{\mathbb{C}_p}/m_{O_{\mathbb{C}_p}}$.

Proof:

1. taking the derivatives of f we have that $\frac{f^{(k)}(x_0)}{k!} = \sum_{n=0}^{+\infty} a_{n+k}\binom{n+k}{k}x_0^n$. Passing to the p-adic valuations we have that $v_p(x_0^n) \geq 0,\ v_p(\binom{n+k}{k}) \geq 0$ and therefore we have that $v_p(\frac{f^{(k)}(x_0)}{k!}) \to +\infty$ and that $\inf\ v_p(\frac{f^{(n)}(x_0)}{n!}) \geq \inf\ v_p(a_n)$
2. $f(x) = \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n(x-x_0+x_0)^n$; now we apply Newton's binomial formula and use the absolute convergence to exchange the summands yielding $\sum_{n=0}^{+\infty} a_n \sum_{k=0}^{+\infty}\binom{n}{k}(x-x_0)^k x_0^{n-k} = \sum_{k=0}^{+\infty}(x-x_0)^k \sum_{n=0}^{+\infty} a_n\binom{n}{k}x_0^{n-k} = \sum_{k=0}^{+\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$. Moreover, taking derivatives by this formula and taking p-adic valuations, we have that $\inf\ v_p(a_n) \geq \inf v_p(\frac{f^{(n)}(x_0)}{n!})$ and using part 1) we have the equality.

3. Clearly $\inf\ v_p(a_n) \leq \inf_{x \in O_{\mathbb{C}_p}} v_p(f(x))$. Since $v_p(a_n) \rightarrow +\infty$ the infimum must be effectively reached at some integer n_0. We may suppose that $v_p(a_{n_0}) = 0$. Let $\bar{f}(T)$ be the reduction of f modulo the maximal ideal $m_{O_{\mathbb{C}_p}}$, then by our hypothesis $\bar{f}$ must be a polynomial with coefficients in $\mathbb{F}_p$ and therefore it has a finite number of roots in $O_{\mathbb{C}_p}/m_{O_{\mathbb{C}_p}}$. Then, if we take x to be an element of $O_{\mathbb{C}_p}$ which does not reduce modulo $m_{O_{\mathbb{C}_p}}$ to a root of $\bar{f}$, it follows that $v_p(f(x)) = 0$ and the claim is proved.

We can now define analytic functions over $\mathbb{C}_p$. If $x_0 \in \mathbb{C}_p,\ r \in \mathbb{R}^+$, we can define the *disc of center* x_0 *and radius* r as the set

$$D(x_0, r) = \{x \in \mathbb{C}_p \mid v_p(x - x_0) \geq r\}. \tag{34}$$

Then a function $f : D(x_0, r) \rightarrow \mathbb{C}_p$ is said to be *analytic* if

$$\lim_{n \rightarrow +\infty} v_p(\frac{f^{(n)}(x_0)}{n!}) + nr = +\infty \tag{35}$$

and consequently define a valuation over the set of analytic functions given by

$$v^r_{x_0}(f) = \inf\ (v_p(\frac{f^{(n)}(x_0)}{n!}) + nr). \tag{36}$$

Proposition: Let $f : D(x_0, r) \rightarrow \mathbb{C}_p$ be an analytic function. Then:

1. $\forall k \in \mathbb{N}, f^{(k)}$ is analytic and $v^r_{x_0}(f^{(k)}) \geq v^r_{x_0}(f)$ and goes to $+\infty$ when $k \rightarrow +\infty$;
2. f can be written as a Taylor power series at every $x \in D(x_0, r)$;
3. $v^r_{x_0}(f) = inf_{x \in D(x_0,r)} v_p(f(x))$.

Proof:(Sketch)

1. Simply compute the derivatives of $f^{(k)}$ from f-th ones and use the definition of analytic.
2. If $r \in \mathbb{Q}$, then choose $\alpha \in \mathbb{C}_p$ with $v_p(\alpha) = r$. Define the auxiliary function $F(x) = f(x_0 + \alpha x)$ and apply to this function the previous lemma to get the result. If $r \notin \mathbb{Q}$ choose a sequence

$(r_n) \in \mathbb{Q}$ which converges to r decreasingly. Using the fact that $D(x_0, r) = \cup D(x_0, r_n)$ we turn back to the rational case and we are done.

3. Use the same technique as 2).