Forschungsseminar on p-adic Galois representations

Organizers: Gebhard Böckle, Gabor Wiese

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Abstract

The aim of this seminar is to give an introduction to Fontaine's theory of p-adic Galois representations of the absolute Galois group G_K a local field K of residue characteristic p. Such representations arise from the cohomology of p-adic étale sheaves on varieties over number fields, when restricting the representation to the decomposition group at p. Concrete examples are the Galois representations attached to modular forms.

One aim of Fontaine's theory is to recover arithmetic information contained in these p-adic representations, for instance the de Rham filtration of de Rham cohomology, or the crystalline Frobenius of crystalline cohomology. Thus geometry motivates the definition of a hierarchy among these representations (Hodge-Tate, de Rham, semistable, crystallline).

It is not possible to cover all the above topics in the seminar. We will mainly concentrate on local fields and their representations and say little about the geometric side. The program follows closely the book manuscript by Fontaine and Ouyang [FO], supplemented by a nice survey of Berger [Be]. Large parts of the seminar will be devoted to studying the infrastructure necessary to state some key results of the theory. The seminar will end with the main theorems proved in [FO]: Theorem A Every de Rham representation is potentially semi-stable. Theorem B There is an equivalence between semistable representations of G_K and filtered **admissible** (φ, N)-modules over K.

- Time: Thursday, 10-12 a.m.
- Place: T03 R04 D10
- Begin: 16 October 2008
- Language: English.
- Webpage (and gateway to bibliography): http://maths.pratum.net/pAdicGR
- Script: Every partipant should type some **text on his talk** containing at least precise definitions and statements of the theorems (that can be done after the talk). That text will be made available on the webpage.

ℓ-adic Galois representations

1 (16/10/2008) ℓ-adic representations of local fields, Hélène Esnault

Level: The talk is not difficult **if** one feels comfortable using words like étale cohomology.

In this talk and in parts of the following talk we survey briefly the much simpler case of ℓ-adic Galois representations before we pass to the p-adic case. [There is a convention regarding the use of ℓ and p that is observed by most, but not all authors.]

Generalities (30 min) This is [FO, §1.1]. Recall Def's 1.1, 1.4, 1.6 and 1.7. Explain why every ℓ-adic Galois representation V is isomorphic to $T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for a $\mathbb{Z}_\ell$-representation T. [Since the notation K_n and K_∞ will have a completely different meaning in later talks, it might be better not to introduce it.] There are obvious constructions (1.1.3) from linear algebra which need to be mentioned. Perhaps this is a good place to introduce $\mathrm{Rep}_{\mathbb{Q}_\ell}(G)$ and discuss [FO, bottom p. 53]. Remind us briefly of the examples in 1.1.4.

ℓ-adic representations of G_K for K finite (30 min) This is [FO, §1.2]. For finite K, an ℓ-representation of G_K is completely characterized by the image of the Frobenius automorphism (1.2.1). [recall geometric versus arithmetic]. Such representations arise naturally from étale cohomology (1.2.2). Define *Weil-number* and *pure of weight* w as in 1.15 and 1.17. In the course of the proof of the Weil-conjectures, Deligne (1.13) showed that $H^m_{\text{ét}}(X, \mathbb{Q}_\ell)$ is pure of weight m for X/K smooth projective and geometrically connected. Recast Theorem 1.13 in terms of étale cohomology: $Z_X = \prod_{i=0}^{2d} P^{(-1)^i}_{H^m_{\text{ét}}(X,\mathbb{Q}_\ell)}$. State conjecture 1.20 and comment on it.

ℓ-adic representations of G_K for K local of residue characteristic $p \neq \ell$ (30 min) This is [FO, §1.3.1]. Recall $P_K \le I_K \le G_K$ and what we know about I_K/P_K. Then introduce and motivate the notation $G_{K,\ell}$ and $P_{K,\ell}$. In this section one can actually prove some things (perhaps Thms 1.24 and 1.26)! The proof of Thm 1.26 is surprisingly simple. If time remains say something about [FO, 1.3.3].

2 (23/10/2008) B-representations and regular G-rings, Ralf Butenuth

Level: Mostly self-contained and rather elementary.

This talk introduces an important method, perhaps introduced by Fontaine (?), to pass from one description of a representation to another one. In this seminar we always want to pass from a Galois representation (via a *period ring B*) to some 'linear algebra type datum'. However, the most classical case is somewhat different: Let X be a smooth projective geometrically connected variety over $\mathbb{Q}$. Via its complex of differentials $\Omega^\bullet_X$, we can attach to X its (algebraic) de Rham cohomology $H^m_{\mathrm{dR}}(X) = \mathbb{H}^m(X, \Omega^\bullet_X)$ (see e.g. [FO, p. 149]). There is an obvious formalism for de Rham cohomology under base extension from $\mathbb{Q}$ to any extension field. At the same time algebraic topology yields the Betti cohomology $H^m_{\mathrm{Be}}(X(\mathbb{C}), \mathbb{Q})$. The de Rham theorem states the isomorphism

$$H^m_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^m_{\mathrm{Be}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Let us fix $\mathbb{Q}$-bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_{n'}\}$ of $H^m_{\mathrm{dR}}(X)$ and $H^m_{\mathrm{Be}}(X)$, respectively. Then the above isomorphism says that $n = n'$ and that there is an invertible matrix $(b_{ij})_{i,j=1}^n \in \mathrm{GL}_n(\mathbb{C})$ expressing the above isomorphism with respect to the chosen bases. The elements $b_{ij} \in \mathbb{C}$ are called *periods*. The matrix (b_{ij}) is not unique, unless we have canonical choices of bases. But: The field $\mathbb{Q}(b_{ij} : i, j = 1, \dots, n)$ is an invariant of X and so is its transcendence degree over $\mathbb{Q}$. The de Rham isomorphism says in particular that $\mathbb{C}$ contains all periods of all smooth projective geometrically connected varieties over any subfield of $\mathbb{C}$.

Explain [FO, 2.1] (60 min) Before proving Prop. 2.6 it might be helpful to recall without proof 0.5.2 on $H^i_{\mathrm{cont}}(G, M)$ for M. Note that in the course of the proof of Thm. 2.13, the following is shown for B-admissible V: If we fix bases $\{e_1, \dots, e_n\}$ of V over F and $\{f_1, \dots, f_n\}$ of $D_B(V)$ over E, then the period matrix defining $B \otimes_F V \cong B \otimes_E D_B(V)$ lies in $\mathrm{GL}_n(B)$. (The inclusions $E, F \subset B$ are part of Def 2.8.) See also [Be, I.2.3].

Recast pst-ℓ-adic representations in the above formalism (30 min) As a first example in the spirit of later examples in this seminar, give the characterization of potentially semistable ℓ-adic representation in terms of the $(\mathbb{Q}_\ell, G_K)$-regular ring B_ℓ as presented in [FO, 1.3.2]. It might be helpful to make B_ℓ with its G_K-action and the differential operator N (matricially) more explicit than done in [FO]. The ring B_ℓ also has a description as $\mathrm{Sym}_{\mathbb{Q}_\ell}(V_\ell(E)(-1))$ modulo a relation $1 = x$ where x is a generator of $\mathbb{Q}_\ell \subset V_\ell(E)(-1)$. If time remains, say something about the conjectures [FO, §1.3.4] (which requires parts of [FO, §1.3.3]).

The theory of (Φ, Γ)-modules

3 (30/10/2008) Mod p Galois representations of G_E with $\operatorname{char} E = p$, Björn Buth

Level: Mostly self-contained.

For a p-adic field K let $K^{\mathrm{cyc}} = \bigcup_n K(\varepsilon^{(n)})$ where $\varepsilon^{(n)}$ is a primitive p^n-th root of unity. The Galois group $\mathrm{Gal}(K^{\mathrm{cyc}}/K)$ is a subgroup of $\mathbb{Z}_p^* \cong \mathbb{F}_p^* \times \mathbb{Z}_p$. Let K_∞ denote its unique subfield with $\mathrm{Gal}(K_\infty/K)$ mapping surjectively to $\mathbb{Z}_p$. Consider

$$1 \longrightarrow H_K = \mathrm{Gal}(K^{\mathrm{alg}}/K_\infty) \longrightarrow G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K) \longrightarrow \Gamma_K = \mathrm{Gal}(K_\infty/K) \longrightarrow 1.$$

It is a key insight (due to Fontaine?) that the group H_K is (canonically) isomorphic to $\mathrm{Gal}(\mathbf{E}_K^{\mathrm{sep}}/\mathbf{E}_K)$ where $\mathbf{E}_K$ is the *field of norms* of Fontaine and Wintenberger (this need not be explained). It satisfies $\operatorname{char} \mathbf{E}_K = p$. So as an intermediate step to understanding p-adic representations of G_K we need to study those of G_E for E a field with $\operatorname{char} E = p$. [Unfortunately [FO] and [Be] use almost disjoint notations. The ring $\mathbf{E}$ in [Be] is different from E in [FO].] Before turning to p-adic representations of G_E one needs to understand mod p representations, since they occur naturally as subquotients of $\mathbb{Z}_p$-representations.

Mod p representations of G_E (60 min) This is [FO, 2.2]. The presentation in [FO] is fairly complete. The key result is Thm. 2.21 which asserts an equivalence of categories between $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ and the category $\mathcal{M}_\varphi^{\text{ét}}(E)$ whose definition is part of the present talk. The equivalence is given by explicitly described functors. Lemma 2.22 is also called Lang's theorem or the Lang torsor, and so one may choose another proof of Lang's theorem (from the literature).

The field $\widehat{\mathcal{E}^{\mathrm{ur}}}$ (30 min) This is a preparation for the following talk. Following [FO, 2.3.1 and 2.3.2] the field $\widehat{\mathcal{E}^{\mathrm{ur}}}$ should be introduced. One can and should say a few words about Cohen rings (e.g. Thms. 0.42 and 0.43) but not too much. The referencing to 0.43 in the 3rd paragraph of 2.3.2, I find confusing. Thms. 2 and 3 and Cor. 1 of [Se, §III.5] seem more to the point.

4 (06/11/2008) p-adic Galois representations of G_E with $\operatorname{char} E = p$ and the ring R, Gebhard Böckle

Level: Rather elementary, depends on Talk 3

The first aim of this lecture is to extend the results of the previous talk to p-adic representations. The second aim is to lay foundations for the desired application of the results for G_E to G_K where K has residue characteristic p and $E = \mathbf{E}_K$ (see the introduction to the previous talk). Note that since $G_E \cong H_K$, the field E is much larger than the residue field of K.

p-adic Galois representations of G_E (30 min) This is [FO, 2.3.3 and 2.3.4]. One should recall the results and notation from the previous talk. The proofs of the main results, Thm. 2.32 and 2.33 seem to be straightforward.

The ring R (60 min) This is [FO, 4.1]. Let C be the completion of a fixed algebraic closure of K. As we shall prove later, it is also the completion of the separable closure of K. Fontaine defines the ring R as $\varprojlim(C/pC \leftarrow C/pC \leftarrow C/pC \leftarrow \ldots)$ with transition map $x \mapsto x^p$. Its elements are denoted by (x_n). It is important to have a second description as $R = \varprojlim(C \leftarrow C \leftarrow C \leftarrow \ldots)$ where again the transition map is $x \mapsto x^p$. This time the elements are denoted by $(x^{(n)})$. The definition of addition via the second limit is not so apparent. However the second description provides a valuation on R. It is shown that $\operatorname{Frac} R$ is complete and algebraically closed. The meaning of R will reveal itself in the next talk. To acquire some familiarity with the notations (x_n) and $(x^{(n)})$ and the conversion between them, it will be good to see a fair number of computations as in [FO, 4.1].

5 (13/11/2008) The action of G_K on $\operatorname{Frac} R$ and (φ, Γ)-modules, Juan Cervino

Level: Rather elementary, depends on Talks 3 and 4.

It is the Galois action on $\operatorname{Frac} R$ that will provide the important link back to K. Once it is understood, the definition of (φ, Γ)-module will be natural, as will be the equivalence of the category of such with that of p-adic Galois representations. Note: The term field of norms does not appear in [FO] and neither its original construction. It is the field $\mathbf{E}_K$ in (4.6).

The action of G_K on R (60 min) This is [FO, 4.2]. From the construction of R it is clear that it carries a continuous action of G_K. In [FO, 4.2.1], for any closed subgroup $H \leq G_K$ the Galois invariants R^H are studied: Associate to an unramified extension K_0 of $\mathbb{Q}_p$ with residue field k the field $E_0 := k((\pi))$ and let K_0^{cyc} be defined as in the introduction to Talk 3. Then for $H = \mathrm{Gal}(K^{\mathrm{alg}}/K^{\mathrm{cyc}})$ it is shown that $(\operatorname{Frac} R)^H$ is identified with the topological completion of the radicial completion of E_0. In [FO, 4.2.2] the fundamental isomorphism

$H_K \to \mathrm{Gal}(E_0^{\mathrm{sep}}/E_0)$ is established. Note that the right hand side is also isomorphic to $\mathrm{Gal}(\mathrm{Frac}\, R/R^H)$. To help our intuition it might be good to quote [FO, Prop. 3.8] whose proof will be given in Talk 6.

Let me say a word about Lem. 4.18: If L/K is a totally ramified extension of local fields and f is an Eisenstein polynomial in $\mathcal{O}_K[x]$ with $L \cong K[x]/f$, then $\delta_{L/K} = \prod_{f(\alpha)=f(\alpha')=0}(\alpha - \alpha')$ where the product is over all pairs of roots of f which are distinct (see 0.73). Since all α are integral but not units, all differences have positive valuation. So if one knows that $v(\mathfrak{D}_{L/K})$ is very close to zero (and hence also $\frac{1}{[L:K]}v(\delta_{L/K})$), then all valuations of all differences are close to zero. This essentially proves that Lem 4.18 is implied by Prop. 0.88 which is covered in Talk 7.

[Note: The assertion of Prop. 4.15 is wrong – the right hand sides of the s.e.s.'s should be $1 + pO_C$ and $1 + \mathfrak{m}_C$, respectively, I believe.]

(φ, Γ)**-modules (30 min)** This is mainly [FO, 4.4]. The aim for the remaining talk is to introduce (φ, Γ)-modules (Def. 4.22) and to prove Thm. 4.23. One should display some of the content and some of the diagrams of [FO, 4.3] but only in as far as they are relevant for Thm. 4.23. In particular, due to the way [FO] introduce E_0, one has to comment on the meaning of $\mathbf{E}_{K_0}$ and on $\mathbf{E}_K$ for general finite extensions $K/\mathbb{Q}_p$ (see [FO, 3.4]).

The theory of Sen and Tate

6 (20/11/2008) The field C, Adam Mohamed

Level: Self contained and rather elementary.

In the previous talks, we have given a description of p-adic Galois representations as modules over $\widehat{\mathcal{E}^{\mathrm{ur}}}$ carrying a continuous action of $\mathbb{Z}_p$. We well see in the next so many talks that those representations that come (or seem to come) 'from geometry' are characterized in a different way. A first natural question (if one thinks of C as the p-adic analog of $\mathbb{C}$) is: Which are those p-adic representations for which C is a period ring (or more precisely, which are those which are C-admissible)? Much of this was question was solved by Sen. The work is somewhat technical. But the pain is worth it, since we will learn some important methods. The first talk will clarify a number of relevant properties of C.

Krasner's Lemma (20 min) This is [FO, 3.1.1]. The lemma should be stated and proved (easy!). It has the following 'well-known' consequence: If K is a complete nonarchimedean field. Then $\widehat{K^{\mathrm{sep}}} \cong \widehat{K^{\mathrm{alg}}}$ and both are algebraically closed.

The Ax-Sen Lemma (40 min) This is [FO, 3.1.2]. Given a finite extension E/K of complete nonarchimedean fields and an element $\alpha \in E$, the Ax-Sen Lemma gives a uniform expression in terms of α of how well it can be approximated by elements of K. One should give the proof of the lemma in characteristic zero and p. As a consequence one should prove Proposition 3.8.

Remedial course on the higher ramification filtrations (30 min) This is in [FO, 0.3], but see also [Se, IV]. We assume some basic knowledge of local fields (by which we mean finite extensions of $\mathbb{Q}_p$). Let L/K be a finite Galois extension of local fields and $G = \mathrm{Gal}(L/K)$. By [Se, III.6.Prop.12], there exists $x \in L$ such that $\mathcal{O}_L = \mathcal{O}_K[x]$. Introduce the function $i_G(s)$ and the higher ramification groups G_i in lower indexing and their basic properties 0.50, 0.51, 0.52 (no proofs), and note that G_0/G_1 is cyclic of order prime to p and G_1 is a p-group. As an important example, compute the higher ramification groups G_i for $\mathrm{Gal}(\mathbb{Q}(\varepsilon^{(n)})/\mathbb{Q})$ for $\varepsilon^{(n)}$, see Talk 3. [FO, Prop. 0.60] asserts that the lower numbering behaves well with respect to passing to subgroups. It does not do so with respect to quotients. Therefore one introduces the upper numbering which behaves well with respect to passing to quotients. Formally one defines the function $\Phi : [-1, \infty) \to [-1, \infty)$ as in (0.18). It is piecewise linear and continuous and starts at $(-1, -1)$. A way to remember its definition is via its slopes: The slope on $(i-1, i)$ is $[G_0 : G_i]^{-1}$. Thus Φ is strictly increasing and hence has an inverse, called Ψ. One defines $G^v := G_{\Psi(v)}$ and where G_u for $u \geq -1$ is defined as $G_{\lceil u \rceil}$. A computation of the higher ramification filtration of the groups $\mathrm{Gal}(\mathbb{Q}(\varepsilon^{(n)})/\mathbb{Q})$ and the deduction of that of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ would be marvelous. Without proof state now Formula [FO, (0.38), p.35].

7 (27/11/2008) C-representations I, Andre Chatzistamatiou

Level: Okay, rather self-contained, depends on Talk 6.

Having understood some basics about C, we next turn to C-representations. But before that, we have a

Remedial course on the different (50 min) The aim is to give a proof of [FO, Prop. 0.88]. To do so first recall [FO, 0.3.5]. Let L/K be a finite *separable* extension of local fields. Then $L \times L \to K : (x, y) \mapsto \mathrm{Tr}(xy)$ is a non-degenerate K-bilinear form on L. Define the different $\mathfrak{D}_{L/K}$ as the inverse of the fractional ideal $\{x \in L \mid \mathrm{Tr}(x\mathcal{O}_K) \subset \mathcal{O}_K\} \supset \mathcal{O}_L$. State the relation between $\mathfrak{D}_{L/K}$ and $\delta_{L/K}$ without proof as well as 0.70, 0.71, 0.72 and 0.73. Now state and prove 0.75 and Cor. 0.76, as well as Prop. 0.88 and Cor. 0.89, and comment on Rem. 0.90.

It should be discussed with the speaker of the following talk whether Prop. 0.91 and Cor 0.92 should be covered rather in this or the following talk.

Almost étale descent (40 min) The aim is the proof of [FO, Prop. 3.12]. Recall why this might be interesting ([FO, Prop. 2.6]). Then give the proof, following [FO, §3.2.1]. If needed, recall some facts about H^1_{cont} from [FO, 0.5].

8 (04/12/2008) C-representations II, Lassina Dembélé

Level: More advanced, depends on Talks 6 and 7.

The aim of this talk is [FO, Prop. 3.16] which for obvious reasons can be called *decompletion.* Together with the previous talk [FO, Thm 3.17] will be immediate. (It might be good to state this at the beginning.)

Tate's normalized Trace map (40 min) This is [FO, 0.91-0.97]. One should make a good selection of the material which to present and which to quote.

Decompletion (40 min) Following [FO, 3.2.2], give the proof of [FO, Prop. 3.16]

Consequences for C-representations (10 min) This is [FO, 3.2.3]. The main task is to state the results. The proofs are really straightforward at this point. (One could also begin the talk with this part assuming Prop. 3.16.)

9 (11/12/2008) Sen's Θ-operator and C-admissible representations, Haluk Sengün

Level: Easier than Talk 8, depends on Talks 6, 7 and 8.

Given a C-representation, we have learned in the previous talk that it descends to a K_∞-representation with an action of $\Gamma_K \cong \mathbb{Z}_p$. Sen's idea is to linearize this action, i.e., to study the induced Lie algebra action. This is his Θ-operator.

Basic properties of Θ (40 min) This is described in [FO, 3.2.4]. I suggest to cover 3.23-3.28 fairly completely and perhaps 3.29 and 3.30 without proof.

The main theorem on Θ (20 min) State and explain the important Theorem 3.31. To give its proof would take another session. (It needs further build up on the upper ramification filtration for p-adic Lie type Galois extensions of local fields.) Deduce Cor. 3.33 from the main theorem and present Rem. 3.34.

Consequences for C-admissible representations (40 min) This is [FO, 3.5]. Consider $K^{\mathrm{alg}} \subset P^{\mathrm{alg}} := \widehat{K^{\mathrm{ur}}}^{\mathrm{alg}} \subset C$. The key results to be presented are: A p-adic representation (ρ, V) of G_K is admissible for K^{alg}, for P^{alg}, for C, respectively, iff $\rho(G_K)$, or $\rho(I_K)$, or $\rho(I_K)$ is finite. The latter two cases are also equivalent to $\Theta = 0$. With our preparations the proofs are fairly straightforward. Moreover one should state Cor. 3.57

While, as a ring of periods, C turned out to be of little use, the methods developed by Sen and the operator Θ are really important!

Crystalline, semistable and de Rham representations

10 (18/12/2008) Witt vectors, Dzmitry Doryn

Level: Survey talk, self-contained.

Before we come to the definition of the rings of periods by Fontaine, it is perhaps a good idea to recall the construction of the Witt vectors and their main properties as described in [FO, 0.2]. Personally, I prefer Serre's approach in [Se, II.4-6], where one first studies p-rings and then follows Lazard to prove basic results on Witt vectors. The speaker can also choose a third approach, e.g. [Ha]. But all results of [FO, 0.2.1-0.2.4] have to be covered in some way. The talk should include Thms. 0.42 and 0.43. The more can be said about 0.2.4, the better. For instance one could try to explain how to find the Cohen ring

$$\Big\{ \sum_{n=-\infty}^{\infty} a_n x^n \mid \forall n : a_n \in \mathbb{Z}_p, \lim_{n\to\infty} a_{-n} = 0 \Big\}$$

with residue field $k := \mathbb{F}_p((x))$ inside $W(k)$.

11 (08/01/2009) The period field B_{dR}, Doan Trung Cuong

Level: Depends on Talks 4, 5 and 10.

In this talk we will introduce the first of Fontaine's period rings, B_{dR} (and its associated graded ring B_{HT}).

The ring B_{HT} (10 min) Following [FO, 5.1], introduce the ring B_{HT} (Def. 5.1) and state and prove Prop. 5.2. The other parts of 5.1 will be treated later.

The ring B_{dR} (55 min) Give a complete presentation of the results and remarks in [FO, 5.2.1-5.2.3]. Not all proofs have to be given. Some representative arguments will be appreciated, since they give the audience an idea on how to compute within $W(R)$ and $B_{\mathrm{dR}}^{(+)}$. The two topologies on B_{dR} are important, as is the fact that B_{dR} is complete with respect to the finer topology.

The cohomology of B_{dR} (25 min) Present [FO, Prop. 5.24] with its proof as well as Prop. 5.25 (note that $H^i = H^i_{\mathrm{cont}}$). The proof of 5.24 may require some work. For instance, the reduction step to a finite extension L/K_∞ in the proof of Prop. 5.24 needs some justification in the case of continuous cohomology (is it true?).

12 (15/01/2009) de Rham representations, Christian Liedtke

Level: Advanced.

de Rham and Hodge-Tate representations (30 min) Follow [FO, 5.1.1, 5.2.5] to give a brief discussion of Hodge-Tate and de Rham representations. In particular, define these terms, define Hodge-Tate numbers, define $\mathbf{Fil}_K$ and state and prove [FO, Thm. 5.28]. Observe [FO, Prop. 5.29] that every deRham representation is Hodge-Tate. Another characterization of being Hodge-Tate is that the operator Θ of Sen is diagonalizable with integral eigenvalues, cf. [Be, II.1.2]. [FO, Prop. 5.30] gives an example of a representation which is not Hodge-Tate. Another perspective on this is given in [Be, II.1.2].

A comparison isomorphism(10 min) The real importance of B_{dR} is explained in [FO, Thm. 5.32], which gives B_{dR} the meaning of a period ring. It might be nice to state the result as well as 5.33, 5.34. The proof of 5.32 is much beyond the scope of the seminar. In Talk 13 we will see the isomorphism of Thm. 5.32 for an elliptic curve with semistable reduction. Here one could mention that the element $t = \log[\varepsilon]$ is a period for the cyclotomic character.

de Rham representations as overconvergent (φ, Γ)-modules (50 min) The content of this part is a survey on the role and meaning of overconvergent (φ, Γ)-modules. Sources are [Be, III.3, IV.1, IV.2] and [FO, 5.3]. The results are due to Berger, Cherbonnier and Colmez, [Be1, CC]. The question is: How can one describe de Rham representations in terms of (φ, Γ)-modules? The tools in its solution are overconvergence and the observation, due to Colmez, that one can axiomatize the Sen-Tate method (almost étale descent and decompletion) so that it applies to other situations.

A (φ,Γ)-module is an étale φ-module over $\mathcal{E}$ with a continuous Γ_K-action, where $\mathcal{E} = \mathbf{A}_K[1/p]$ and $\mathbf{A}_K = \{\sum_{n=-\infty}^{\infty} a_n[\pi_K]^n \mid \forall n : a_n \in \mathcal{O}_F, \lim_{n\to\infty} a_{-n} = 0\}$ is the Cohen ring of $\mathbf{E}_K$ and F is the maximal subextension of K which is unramified over $\mathbb{Q}_p$. Thinking of $\sum_{n=-\infty}^{\infty} a_n X^n$ as a Laurent series on C, it follows that its principal part converges on $|X| \geq 1$ and its power series part on $|X| < 1$. The overconvergent ring $\mathbf{A}_K^{\dagger,r}$ is defined so that the principal part converges for $|X| \geq p^{-1/re_K}$ (while still all a_n lie in $\mathbb{Z}_p$) and $\mathbf{A}_K^{\dagger} := \cup_{r>0}\mathbf{A}_K^{\dagger,r}$. The main result of [CC] is that there is an isomorphism between (φ,Γ)-modules and overconvergent (φ,Γ)-modules, where the latter are defined by replacing $\mathbf{A}_K[1/p]$ by $\mathbf{A}_K^{\dagger}[1/p]$. The proof is an adaption of the Sen-Tate method, cf. [FO, Thm.5.55].

Now Berger goes further with the method. He wants to study the differential operator of Sen in this theory. The problem is that not even on the trivial object $\mathbf{A}_K^{\dagger}[1/p]$ this operator is defined. The reason is that elements of $\mathbf{A}_K^{\dagger}[1/p]$ have bounded coefficients, a property not preserved by Sen's differentiation of power series. Berger's solution is to enlarge $\mathbf{A}_K^{\dagger,r}$ to $\mathbf{A}_{\mathrm{rig},K}^{\dagger,r}$, where the latter ring consists of Laurent series with coefficients in F which converge on $p^{-1/re_K} \leq |X| < 1$. On the resulting 'rigid' (φ,Γ)-modules the operator $\partial := \frac{1}{\log(1+[\pi_K])}\Theta$ is well-defined except for poles at '$X = \varepsilon^{(n)} - 1$'. Berger's key theorem is: The de Rham representations V are precisely those for which these poles for the corresponding rigid (φ,Γ)-modules are 'resolvable' by passing to a suitable subobject $N_{\mathrm{dR}}(V) \subset D_{\mathrm{rig}}^{\dagger}(V)$, cf. [Be1].

13 (22/01/2009) The period rings B_{cris} and B_{st}, Stefan Kukulies

Level: Advanced.

The definitions of B_{cris} and B_{st} (40 min) Following [FO, 6.1.1] define A_{cris} and identify it as a subring of B_{dR}^{+}. The latter should be well-explained since A_{cris} is a p-adic completion of A_{cris}^{0} while B_{dR}^{+} was obtained by completing $W(R)[1/p]$ with respect to the principal ideal which was the kernel of the surjection $\theta : W(R)[1/p] \to C$. Props. 6.4, 6.5 and 6.6 describe basic properties of A_{cris}. Their proofs may further our understanding of the structure of A_{cris}. Then define B_{cris}^{+} and B_{cris}. Unlike for B_{dR}, there is a natural Frobenius endomorphism on B_{cris} defined in [FO, 6.1.2]. Then [FO, 6.1.3] gives the steps to define a logarithm $U_R^{+} \to B_{\mathrm{cris}}$. To extend it to $(\mathrm{Frac}\, R)^{*} \to$ one needs to introduce B_{st}, cf. Prop. 6.11. If it is not too technical, one might attempt to give the proof of 6.11. Then state the remaining properties of B_{st} as given in [FO, 6.1.4].

The fundamental exact sequence relating B_{cris} and B_{dR} (25 min) The material [FO, 6.2], which stems from [Fo], is quite technical. Its main purpose is to prove Thm. 6.26. I suggest not to give this proof, but to explain the meaning of a few of the intermediate and the final result. The main problem of the matter seems to be that the filtration on B_{cris} coming from B_{dR} is highly incompatible with the Frobenius φ, cf. [Be, II.3.4]. Here are some suggestions: Introduce the $T^{\{n\}}$, define Λ and explain Thm. 6.21. Another important intermediate result is Prop. 6.24. which one should take on faith. Perhaps building on this, one can indicate (some) proofs (and the meaning) of Thm 6.25 and Thm. 6.26. (see also Rem. 6.27)

Examples (25 min) Given a p-adic representation V of G_K, one can associate to it $D_?(V) := (B_? \otimes_K V)^{G_K}$ for $? \in \{\mathrm{dR}, \mathrm{cris}, \mathrm{st}\}$. If $B_?$ has some additional structure which commutes with G_K this structure is passed on to $D_?(V)$. An instructive example is the case where V arises from the Tate-module of an elliptic curve with semistable reduction. Following [Be, II.4] this example should be discussed in some detail. As Berger says, it was this example which motivated Fontaine to define B_{st} as $B_{\mathrm{cris}}[\log[\varpi]]$. The example computes a period matrix of V for B_{dR} and B_{st}.

14 (29/01/2009) Semi-stable representations and filtered (φ, N)-modules, Gabor Wiese

Level: Okay.

Semistable p-adic representations (10 min) Following [FO, 6.3] it is straightforward to define the notion semistable and crystalline representation. It is obvious that any crystalline representation is semistable. The only result of this section where something needs to be proved in Prop. 6.31 which asserts that semistable representations are de Rham.

(Admissible) filtered (φ, N)-modules (40 min) This is [FO, 6.4]. Once the definition of filtered (φ, N)-module is given, it is straightforward to see that D_{st} maps to the category of filtered (φ, N)-modules. An important question is whether one can characterize the image!

A subcategory of the category of filtered (φ, N)-modules is that of admissible filtered (φ, N)-modules. Its definition is in terms of Newton and Hodge numbers (or polygons). This subcategory is abelian, cf. Prop. 6.50. It was shown by Totaro in [To] (following work of Faltings in the crystalline case) that the category is Tannakian (i.e. that it has duals and a tensor structure). The theorem of Colmez-Fontaine shows that, in fact, the functor D_{st} defines an equivalence of categories between semistable p-adic representations of G_K and admissible filtered (φ, N)-modules over K. This is **Theorem B** in [FO, 6.5.2] whose proof occupies a large portion of [FO, Ch. 7]. At this point, following [FO, 6.4] the basic properties of $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ should be presented, and Theorem B should be stated.

Examples (40 min) To gain more familiarity with the objects of $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$, I recommend at this point to present as much as possible of [FO, 7.1]. (Admissibility in terms of Newton and Hodge polygons and examples in dimension 1 and 2). If more examples are wanted one can also consult [GM] (where however the coefficient field is a finite extension of $\mathbb{Q}_p$).

15 (05/02/2009) Main Theorems, Kay Rülling

Level: Advanced.

Theorem B1 (45 min) Following [FO, 7.2], a proof of Theorem B1 which is the simpler part of the equivalence stated in Theorem B should be attempted. Once the book manuscript [FO, Ch.7] will be complete, the interested audience of the seminar may finish the proof of B in independent study.

Theorem A (45 min) There is a second important theorem clarifying further the relations among the types of representations we have encountered in the seminar. This is **Theorem A** of [FO, p.193]. It says that any de Rham representation over K becomes semistable over a finite extension K' of K. (Or in other words, every de Rham representation is potentially semistable). So perhaps one can start out with a few basic facts from [FO, 6.5.1]. Most of the time could be dedicated to complete the survey begun in Talk 12, following [Be, IV.3 and IV.5]:

The approach (again due to Berger) is to express crystalline and semistable representations in terms of rigid (and hence in particular overconvergent) (φ, Γ)-modules. Berger proves that $D_{\mathrm{cris}}(V) \cong (D^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$ (and the variant $D_{\mathrm{st}}(V) \cong (D^{\dagger}_{\mathrm{rig}}(V)[1/t, \log[\varpi]])^{\Gamma_K}$). There are some hints about why this works in [Be, IV.3.3] and some motivational words on how to imagine all the rings $B^{*}_{?}$ in [Be, IV.3.2]. This gives rigid (φ, Γ)-modules a central role in the entire theory. They describe all p-adic Galois representations of G_K and at the same time, for these it is 'easy' to characterize the subclasses of de Rham, semistable and crystalline representations.

At this point, we remember that a de Rham representation gave rise to a module $N_{\mathrm{dR}}(V)$ together with a differential operator ∂. Any potentially semistable representation is de Rham and if V is semistable then ∂ on $N_{\mathrm{dR}}(V)$ turns out to be unipotent. At this point the question is transferred to a conjecture in the theory of p-adic differential equations, namely Crew's conjecture. It says that any ∂ is quasi-unipotent, i.e., that it becomes unipotent after a finite extension of K. Thus Crew's conjecture implies Theorem A. As it turns out, around the time of Berger's 'translation', Crew's conjecture was proved independently and simultaneously by André, Kedlaya and Mebkhout, [An, Ke, Me]. (Note the analogy between Crew's conjecture and Grothendieck's theorem of Talk 1 on potential semistability!).

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