SHIMURA CURVES III

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Introduction. These are the notes of a talk I gave in the Arithmetic Geometry Seminar at the University of Essen on 10th of July 2008. The subject is the Eichler-Shimura isomorphism for quaternionic automorphic forms after Saito (see [Sa06]). We also used the lecture notes of van den Bogaart [Bo05] on the same subject as well as the article [BaNe81].

The aim of these notes is to define quaternionic automorphic forms attached to a quaternion algebra B, to interpret them as global sections of a certain locally free sheaf on the Shimura curve $M(\mathbb{C})$ defined by B and to show that these sections together with their complex conjugates form the Hodge decomposition of a certain local system on $M(\mathbb{C})$. For the definition of $M(\mathbb{C})$ see Stefan's talk [Ku08], for the use of the Eichler-Shimura isomorphism see Garbor's talk [Wi08].

I am anything but a specialist in the field, therefore the reader should be aware of mistakes or wrong statements I might give, which (if there are any) are of course entirely due to me.

Quaternionic Automorphic Forms. For the rest of this notes we fix the following notations.

- $F/\mathbb{Q}$ is a totally real number field, $[F:\mathbb{Q}] = n$, $I = \{\tau_1, \ldots, \tau_n\} = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$. We view $F \subset \mathbb{R}$ via τ_1, which is fixed. If n is even we fix a finite place v of F.
- $\mathbb{A}_f$ are the finite adèles of $\mathbb{Q}$.
- B is a quaternion algebra, which ramifies exactly at $\{\tau_2, \ldots, \tau_n, v\}$ (i.e. $B \otimes_F F_w$ is a division algebra for $w \in \{\tau_2, \ldots, \tau_n, v\}$). This property determines B uniquely up to isomorphism.
- Let $G := \mathrm{Res}_{F/\mathbb{Q}} B^\times$ be the Weil restriction of the algebraic group $B^\times$ to $\mathbb{Q}$, in particular $G(A) = (B \otimes_{\mathbb{Q}} A)^\times$, for A a $\mathbb{Q}$-algebra. We have
$$G(\mathbb{R}) = \mathrm{Gl}_2(\mathbb{R}) \times (\mathbb{H}^\times)^{n-1}$$
and
$$G(\mathbb{R})_+ = Gl_2(\mathbb{R})_+ \times (\mathbb{H}^\times)^{n-1}, \quad G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+.$$
- $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m\mathbb{C}})$.
- $$h : \mathbb{S}(R) = \mathbb{C}^\times \to \mathbb{G}(\mathbb{R}), \quad z = x + iy \mapsto h(z) = \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, 1 \ldots, 1 \right).$$
Denote $X = \{gh(-)g^{-1} | g \in G(\mathbb{R})\}$. Then (see Stefan)
$$X \xrightarrow{\simeq} \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}), \quad gh(-)g^{-1} \mapsto gi = \frac{ai+b}{ci+d},$$

with $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a_2 \dots a_n \right) \in G(\mathbb{R})$.

- Let X^+ be the connected component of h, it is isomorphic to the Poincaré upper half plane.
- Let $k = (k_1, \dots, k_n, w)$ be a multi weight with $w \geq k_j \geq 2$, $k_j \equiv w \mod 2$.

$$m := \prod_{j=1}^{n} (k_j - 1).$$

Set

$$V_{\mathbb{C}} := \mathbb{C}^{\oplus 2}, \quad e_0 := (1,0), e_1 := (0,1) \in V_{\mathbb{C}}^{\vee} = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

and

$$W_{j,\mathbb{C}} := \mathrm{Sym}^{k_j - 2}(V_{\mathbb{C}}^{\vee}), \quad j = 2, \dots, n, \quad W' := W_2 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} W_n.$$

Notice that $\{e_0^{r_2} e_1^{s_2} \otimes \dots \otimes e_0^{r_n} e_1^{s_n} \mid r_j + s_j = k_j - 2,\, 2 \leq j \leq n\}$ is a basis for $W'_{\mathbb{C}}$. (The choice of the strange notation will become apparent later. Finally it is done in such a way, that it harmonizes with Saito's definitions whose choice is dictated by the wish to obtain a Hodge theoretic description of the quaternionic automorphic forms we are going to define, with conventions as in [De79] and not as in [De71].)

Definition 1. (i) A map $f : X \times G(\mathbb{A}_f) \to \mathbb{C}^r$, $r \geq 1$, is *holomorphic* if $f(-, g) : X \to \mathbb{C}^r$ is holomorphic for each fixed $g \in G(\mathbb{A}_f)$ and the map $G(\mathbb{A}_f) \to H^0(X, \mathcal{O}_X^r) = \mathrm{Hol}(X, \mathbb{C}^r)$ is locally constant.

(ii) We define an action of $G(\mathbb{Q})$ on

$$\mathrm{Hol}(X \times G(\mathbb{A}_f), W'_{\mathbb{C}}) \cong \mathrm{Hol}(X \times G(\mathbb{A}_f), \mathbb{C}^{\frac{m}{(k_1-1)}})$$

in the following way: Any element $f \in \mathrm{Hol}(X \times G(\mathbb{A}_f), W'_{\mathbb{C}})$ can uniquely be written in the following form

$$f(z,g) = \sum e_0^{r_2} e_1^{s_2} \otimes \dots \otimes e_0^{r_n} e_1^{s_n} f_{r_2 s_2 \dots r_n s_n}(z,g), \tag{1}$$

where the sum is over all tuples $(r_2, s_2, \dots r_n, s_n)$ with $r_j + s_j = k_j - 2$. Now take $\gamma \in G(\mathbb{Q}) \subset G(\mathbb{C}) \cong \mathrm{Gl}_2(\mathbb{C})^I$ and view it as a tuple of invertible matrices $\gamma = (\gamma_1, \dots, \gamma_n)$, with $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for f as in (1) we define

$$\gamma.f(z,g) := \frac{\det \gamma_1^{\frac{w+k_1-2}{2}}}{(cz+d)^{k_1}} \prod_{j=2}^{n} \det(\gamma_j)^{\frac{w-k_j}{2}} \cdot \tag{2}$$
$$\sum (e_0\gamma_2)^{r_2} (e_1\gamma_2)^{s_2} \otimes \dots \otimes (e_0\gamma_n)^{r_n} (e_1\gamma_n)^{s_n} f_{r_2 s_2 \dots r_n s_n}(\gamma_1 z, \gamma g).$$

(iii) We define a $G(\mathbb{A}_f)$-action on $H^0(X, \mathcal{O}_X \otimes W'_{\mathbb{C}})$ via

$$f(z,g).g' := f(z, gg'), \quad \text{for } z \in X, g, g' \in G(\mathbb{A}_f). \tag{3}$$

(iv) Let $K \subset G(\mathbb{A}_f)$ be open and compact. Then we say $f \in \mathrm{Hol}(X \times G(\mathbb{A}_f), W'_{\mathbb{C}})$ is a *quaternionic automorphic form of level* K *and weight* k if

$$\gamma.f(z,g) = f(z,g) \quad \forall\, \gamma \in G(\mathbb{Q}) \quad \text{and} \quad f(z,g).g' = f(z,g) \quad \forall\, g' \in K.$$

We write $QM_K^{(k)}$ for the $\mathbb{C}$-vector space of all quaternionic automorphic forms of level K and weight k and $QM^{(k)} = \varinjlim_K QM_K^{(k)}$.

Shimura Curves. We recall the definition and some basic facts about Shimura curves.

- $Z_s(G) := \mathrm{Ker}(Z(G) \overset{\mathrm{Nm}}{\to} \mathbb{G}_{m,\mathbb{Q}})$, where $Z(G) = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ is the center of G and on the A-valued points (with A a $\mathbb{Q}$-algebra) Nm is given by the usual norm $\mathrm{Nm} : (F \otimes_{\mathbb{Q}} A)^\times \to A^\times$. In particular $Z_s(G)(\mathbb{Q}) = \mathrm{Ker}(\mathrm{Nm} : F^\times \to \mathbb{Q}^\times)$.
- $G^c := G/Z_s(G)$. Thus $G^c(\mathbb{Q}) = F^\times/\mathrm{Ker}(F^\times \to \mathbb{Q}^\times) \cong \mathbb{Q}^\times$ which is discrete in $G^c(\mathbb{A}_f)$. $G^c(\mathbb{Q}) = B^\times/\mathrm{Ker}(\mathrm{Nm} : F^\times \to \mathbb{Q}^\times)$. Notice that $G^c(\mathbb{Q})$ has (as $G(\mathbb{Q})$) no quasi-unipotent elements, since F is totally real and $B^\times$ is a division algebra.
- For $K \subset G(\mathbb{A}_f)$ open compact, we denote by K^c its image in $G^c(\mathbb{A}_f)$.

Definition 2. We say that an open compact subset $K \subset G(\mathbb{A}_f)$ is *small enough* if

$$\Gamma_g := \frac{gKg^{-1} \cap G(\mathbb{Q})_+}{gKg^{-1} \cap Z(G_+)(\mathbb{Q})} \text{ has no torsion } \forall g \in G(\mathbb{A}_f)$$

and

$$K^c \cap Z(G^c)(\mathbb{Q}) = \{1\} \quad (\Leftrightarrow K \cap Z(G)(\mathbb{Q}) = Z_s(G)(\mathbb{Q})).$$

We notice that $K \subset G(\mathbb{A}_f)$ small enough exist and we have

$$\Gamma_g = gK^cg^{-1} \cap G^c(\mathbb{Q})_+, \quad \forall\, g \in G(\mathbb{A}_f) \text{ and } K \subset G(\mathbb{A}_f) \text{ small enough.} \tag{4}$$

From now on $K \subset G(\mathbb{A}_f)$ will always be an open compact subset, which is small enough.

We recall some facts from Stefan's notes [Ku08]

- The Shimura curve associated to (G, X, K) is defined by
$$M_K(\mathbb{C}) := M_K(G, X) := G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K),$$
where the action of $G(\mathbb{Q})$ on $(X \times G(\mathbb{A}_f)/K)$ is the natural one.
- $$M_K(\mathbb{C}) \cong \coprod_{g \in G(\mathbb{Q})_+ \setminus G(\mathbb{A}_f)/K} \Gamma_g \setminus X^+.$$
- The spaces $\Gamma_g \setminus X^+$ are Riemann surfaces. Since Γ_g has no torsion, the action of Γ_g on X^+ is free. Hence $X^+ \to \Gamma_g \setminus X^+$ is the universal covering and $\pi_1(\Gamma_g \setminus X^+) = \Gamma_g$.
- $\Gamma_g \setminus X^+$ is compact.
- The inclusions $K' \subset K$ give natural maps $M_{K'}(\mathbb{C}) \to M_K(\mathbb{C})$, which are finite. We obtain a projective system $(M_K(\mathbb{C})_K)$ and the Shimura curve associated to (G, X) is then defined by
$$M(\mathbb{C}) := M(G, X) := \varprojlim_K M_K(\mathbb{C}).$$
It is a scheme over $\mathbb{C}$, with $\mathbb{C}$-valued points given by
$$M(\mathbb{C}) = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/\overline{Z(G)(\mathbb{Q})}),$$
where $\overline{Z(G)(\mathbb{Q})}$ is the closure of $Z(G)(\mathbb{Q})$ in $Z(\mathbb{A}_f)(\mathbb{Q})$ (see [Mi90]). We denote by M the canonical model over F (see [De79], [Mi90].)

Remark 3 (see [BaNe81])**.** Recall that for a ringed topological space $(Y, \mathcal{O}_Y)$, on which a group Γ acts from the left, and $\mathcal{F}$ an $\mathcal{O}_Y$-sheaf on Y, one says that Γ *acts on* $\mathcal{F}$ or $\mathcal{F}$ *is a* Γ*-sheaf* if for all $\gamma \in \Gamma$ one has an isomorphism $\mathcal{F} \xrightarrow{\simeq} \gamma_* \mathcal{F}$, which is compatible with the group structure. If $\mathcal{F}$ is Γ-sheaf one obtains a sheaf on the quotient $\Gamma \setminus Y$ by taking Γ-invariants, $\mathcal{F}^\Gamma$. More concretely if $\pi : Y \to \Gamma \setminus Y$ is the quotient map, then $\mathcal{F}^\Gamma$ is defined as follows

$$\Gamma \setminus Y \supset U \mapsto \mathcal{F}(\pi^{-1}(U))^\Gamma.$$

If Γ acts freely on Y, then

$$(\mathcal{F}^\Gamma)_{\pi(y)} = \mathcal{F}_y.$$

In particular $\mathcal{F}^\Gamma$ is locally free and of finite rank if $\mathcal{F}$ is. Furthermore if Γ acts freely on Y and $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of finite rank on Y, then

$$(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})^\Gamma = \mathcal{F}^\Gamma \otimes_{\mathcal{O}_{\Gamma \setminus Y}} \mathcal{G}^\Gamma, \quad (\mathrm{Sym}^n_{\mathcal{O}_Y} \mathcal{F})^\Gamma = (\mathrm{Sym}^n_{\mathcal{O}_{\Gamma\setminus Y}} \mathcal{F}^\Gamma).$$

The Eichler-Shimura Isomorphism. We give a description of quaternionic automorphic forms as sections of certain locally free sheaves on $M(\mathbb{C})$ and show that $QM^{(k)} \oplus \overline{QM^{(k)}}$ is the Hodge decomposition of a certain local system on $M(\mathbb{C})$. In fact there is a way to make sense of this even over the completion at some prime of a certain number field containing F. We give some hints towards this, see [Sa06] for details.

For the rest of this notes we fix a finite Galois extension L/F, which splits B, i.e. $B \otimes_F L \cong M_2(L)^I$. In particular $G_L \cong Gl_{2,L}^I$ as algebraic groups.

We set

$$V := L^{\oplus 2}, \quad e_0 := (1, 0), e_1 := (0, 1) \in V^\vee = \mathrm{Hom}_L(V, L).$$

Write $\mathbb{P}^1_\mathbb{C} = \mathrm{Proj}\, \mathbb{C}[x_0, x_1]$ and $z = \frac{x_0}{x_1}$ in any neighborhood with $x_1 \neq 0$. $Gl_2(\mathbb{C})$ acts on $\mathcal{O}_{\mathbb{P}^1_\mathbb{C}}$ via

$$g.f(x_0, x_1) = f(ax_0 + bx_1, cx_0 + dx_1), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathbb{C}). \tag{5}$$

Notice that the center of $Gl_2(\mathbb{C})$ acts trivially. We obtain an action $Gl_2(\mathbb{C}) \to \mathrm{Aut}_\mathbb{C}(\mathcal{O}_{\mathbb{P}^1} \otimes_\mathbb{C} V^\vee)$ via

$$g.(f \otimes e_j) = g.f \otimes e_j g, \quad j = 0, 1. \tag{6}$$

We have the following exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \otimes_\mathbb{C} V^\vee \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0, \tag{7}$$

where the map on the left is given by $\varphi \mapsto \varphi(x_1) \otimes e_0 - \varphi(x_0) \otimes e_1$ and the map on the right by $1 \otimes e_j \mapsto x_j$. In any neighborhood U with $x_1 \neq 0$ we can identify

$$\mathcal{O}_U(-1) = \mathcal{O}_U \cdot (e_0 - ze_1) \subset \mathcal{O}_{\mathbb{P}^1} \otimes V^\vee. \tag{8}$$

With respect to the action defined in (6) $e_0 - ze_1$ behaves as follows ($g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

$$g.(e_0 - ze_1) = (e_0 g - (gz) e_1 g) = \frac{\det g}{cz + d}(e_0 - ze_1). \tag{9}$$

Set

$$W_j := \mathrm{Sym}_l^{k_j - 2}(V^\vee), \quad W := W_1 \otimes \ldots \otimes W_n. \tag{10}$$

Notice that $\{e_0^{r_1}e_1^{s_1} \otimes \ldots \otimes e_0^{r_n}e_1^{s_n} \mid r_j + s_j = k_j - 2, 1 \le j \le n\}$ is a basis for W.

Definition 4. We define a G_L^c-representation

$$\rho^{(k)} : G_L^c \longrightarrow Gl(W) \tag{11}$$

in the following way: compose the isomorphism

$$G_L \cong Gl(V)^I \tag{12}$$

with

$$\tilde{\rho}^{(k)} := \bigotimes_{j \in I} \left(\mathrm{Sym}^{k_j - 2} \otimes \det^{\frac{w-k_j}{2}} \right) \circ \check{\mathrm{pr}}_j : Gl(V)^I \to Gl(W),$$

where $\check{\mathrm{pr}}_j$ is the contragredient representation of the j-th projection $Gl(V)^I \to Gl(V)$. Explicitely:

$$\tilde{\rho}^{(k)}(g = (g_1, \ldots, g_n))(e_0^{r_1}e_1^{s_1} \otimes \ldots \otimes e_0^{r_n}e_1^{s_n}) = \left(\prod_{j=1}^{n} \det(g_j^{-1})^{\frac{w-k_j}{2}} \right) (e_0 g_1^{-1})^{r_1}(e_1 g_1^{-1})^{s_1} \otimes \ldots \otimes (e_0 g_n^{-1})^{r_n}(e_1 g_n^{-1})^{s_n}. \tag{13}$$

Notice that (12) composed with $\tilde{\rho}^{(k)}$ restricted to $Z(G_L) \subset G_L$ acts as $\mathrm{Nm}_{F/Q}^{-(w-2)}$ and hence this operation factors to give (11).

Definition 5. Set

$$\mathcal{W}_{\mathbb{P}^1_{\mathbb{C}}} := \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \otimes_{\mathbb{C}} W_{\mathbb{C}} = \mathrm{Sym}_{\mathbb{C}}^{k_1-2}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \otimes_{\mathbb{C}} V_{\mathbb{C}}^{\vee}) \otimes W_{2,\mathbb{C}} \otimes \ldots \otimes W_{n,\mathbb{C}}.$$

We define the following action

$$R^{(k)} : G_{\mathbb{C}}^c \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{W}_{\mathbb{P}^1_{\mathbb{C}}})$$

as the composition of $G_{\mathbb{C}} \cong Gl_2(\mathbb{C})^I$ with

$$\tilde{R}^{(k)} : Gl_2(\mathbb{C})^I \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{W}_{\mathbb{P}^1_{\mathbb{C}}})$$

given by

$$\tilde{R}^{(k)}(g = (g_1, \ldots, g_n))(f \otimes w) := (g_1^{-1}.f) \otimes \tilde{\rho}^{(k)}(g)(w),$$

where $g_1^{-1}.f$ is defined in (5) and $\tilde{\rho}^{(k)}(g)(w)$ is defined above. This composition factors to give $R^{(k)}$.

Definition 6. Set $\alpha := \frac{w-k_1}{2}$. The exact sequence (7) defines a filtration

$$\mathcal{W}_{\mathbb{P}^1_{\mathbb{C}}} = F^{\alpha}_{\mathbb{P}^1_{\mathbb{C}}} \supset F^{\alpha+1}_{\mathbb{P}^1_{\mathbb{C}}} \supset \ldots \supset F^{\alpha+k_1-2}_{\mathbb{P}^1_{\mathbb{C}}} \supset 0$$

with

$$F^{\alpha+p}_{\mathbb{P}^1_{\mathbb{C}}} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)^{\otimes p} \cdot \mathrm{Sym}^{k_1-2-p}(V_{\mathbb{C}}^{\vee}) \otimes W_{2,\mathbb{C}} \otimes \ldots \otimes W_{n,\mathbb{C}} \tag{14}$$

and

$$\frac{F^{\alpha+p}_{\mathbb{P}^1_{\mathbb{C}}}}{F^{\alpha+p+1}_{\mathbb{P}^1_{\mathbb{C}}}} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)^{\otimes k_1-2-p}(V_{\mathbb{C}}^{\vee}) \otimes W_{2,\mathbb{C}} \otimes \ldots \otimes W_{n,\mathbb{C}}. \tag{15}$$

Definition 7.

- For $K \subset G(\mathbb{A}_f)$ small enough and $g \in G(\mathbb{A}_f)$ define
$$\mathcal{W}^{(k)}_{\Gamma_g(K)} := (\mathcal{W}_{\mathbb{P}^1(\mathbb{C})|X^+})^{\Gamma_g(K)} \quad \text{on } \Gamma_g(K) \setminus X^+,$$
where $\Gamma_g(K)$ acts via
$$\Gamma_g(K) \subset G^c(\mathbb{Q})_+ \subset G^c(\mathbb{C}) \stackrel{R^{(k)}}{\to} \mathrm{Aut}_{\mathbb{C}}(\mathcal{W}_{\mathbb{P}^1_{\mathbb{C}}}).$$
- Define
$$\mathcal{W}^{(k)}_K := \prod_{g \in G(\mathbb{Q})_+ \setminus G(\mathbb{A}_f)_+/K} j_{g*}\mathcal{W}^{(k)}_{\Gamma_g(K)},$$
where $j_g : \Gamma_g(K) \setminus X^+ \hookrightarrow M_K(\mathbb{C})$ is the natural inclusion.
- Define
$$\mathcal{W}^{(k)} = \varinjlim_K \pi_K^{-1}\mathcal{W}^{(k)}_K,$$
where $\pi_K : M(\mathbb{C}) \to M_K(\mathbb{C})$ is the projection.
- Define a $G(\mathbb{A}_f)$-action on $\mathcal{W}^{(k)}$ in the following way: For $a \in G(\mathbb{A}_f)$ we have $\Gamma_g(K) = \Gamma_{ga}(a^{-1}Ka)$. Thus
$$\Gamma_g(K) \setminus X^+ \cong \Gamma_{ga}(a^{-1}Ka) \setminus X^+ \quad \text{and} \quad \mathcal{W}^{(k)}_{\Gamma_g(K)} \cong \mathcal{W}^{(k)}_{\Gamma_{ga}(a^{-1}Ka)}.$$
This induces the $G(\mathbb{A}_f)$-action (a sends an element from $\mathcal{W}^{(k)}_{\Gamma_g(K)}$ via the second isomorphism to an element in $\mathcal{W}^{(k)}_{\Gamma_{ga}(a^{-1}Ka)}$.)
- In the same way the trivial connection
$$d : \mathcal{W}^{(k)}_{\mathbb{P}^1_{\mathbb{C}}} \to \mathcal{W}^{(k)}_{\mathbb{P}^1_{\mathbb{C}}} \otimes \Omega^1_{\mathbb{P}^1_{\mathbb{C}}}$$
descends to give connections
$$\nabla_{\Gamma_g} : \mathcal{W}^{(k)}_{\Gamma_g} \to \mathcal{W}^{(k)}_{\Gamma_g} \otimes \Omega^1_{\Gamma_g \setminus X^+},$$
$$\nabla_K : \mathcal{W}^{(k)}_K \to \mathcal{W}^{(k)}_K \otimes \Omega^1_{M_K(\mathbb{C})}$$
and
$$\nabla : \mathcal{W}^{(k)} \to \mathcal{W}^{(k)} \otimes \Omega^1_{M(\mathbb{C})}.$$
This last connection being compatible with the $G(\mathbb{A}_f)$-action.
- In the same way the filtration $F^{\cdot}_{\mathbb{P}^1_{\mathbb{C}}} \subset \mathcal{W}^{(k)}_{\mathbb{P}^1_{\mathbb{C}}}$ descends to give filtrations
$$\mathcal{W}^{(k)}_{\Gamma_g} = F^{\alpha}_{\Gamma_g} \supset \ldots \supset F^{\alpha+k_1-2}_{\Gamma_g} \supset 0,$$
$$\mathcal{W}^{(k)}_K = F^{\alpha}_K \supset \ldots \supset F^{\alpha+k_1-2}_K \supset 0$$
and
$$\mathcal{W}^{(k)} = F^{\alpha} \supset \ldots \supset F^{\alpha+k_1-2} \supset 0.$$
This last being compatible with the $G(\mathbb{A}_f)$-action.

The last part of the filtration becomes an extra name:
$$\mathcal{V}^{(k)}_{\Gamma_g} := F^{\alpha+k_1-2} = \left((\mathcal{O}_{X^+}(e_0 - ze_1))^{\otimes k_1-2} \otimes_{\mathbb{C}} W_{2,\mathbb{C}} \otimes \ldots \otimes W_{n,\mathbb{C}}\right)^{\Gamma_g},$$
$$\mathcal{V}^{(k)}_K := F^{\alpha+k_1-2}, \quad \mathcal{V}^{(k)} := F^{\alpha+k_1-2}.$$

- Considering W as a constant sheaf on $\mathbb{P}^1_{\mathbb{C}}$ we define
$$\mathcal{P}^{(k)}_{L,\Gamma_g} := \left(W_{\mathbb{P}^1(\mathbb{C})_{|X^+}}\right)^{\Gamma_g},$$
where Γ_g acts via
$$\Gamma_g \subset G^c(\mathbb{Q})_+ \subset G^c(L) \stackrel{\rho^{(k)}}{\to} Gl(W).$$
In the same way as above we obtain $\mathcal{P}^{(k)}_{L,K}$ and $\mathcal{P}^{(k)}_{L}$. We set
$$\mathcal{P}^{(k)}_{\mathbb{C}} := \mathcal{P}^{(k)}_{L} \otimes_L \mathbb{C}, \quad \text{etc.}$$

Proposition 8. *Take $K \subset G(\mathbb{A}_f)$ small enough and $\Gamma_g = \Gamma_g(K)$.*

(i) *The sheaves $\mathcal{W}^{(k)}_{\Gamma_g}$, $F^{\alpha+p}_{\Gamma_g}$, $p = 0, \dots, k_1 - 2$, are locally free on $\Gamma_g \setminus X_+$ and $\mathcal{P}^{(k)}_{L,\Gamma_g}$ is a local system.*

(ii)
$$\mathcal{P}^{(k)}_{\mathbb{C}} = (\mathcal{W}^{(k)})^{\nabla} = Ker(\nabla : \mathcal{W}^{(k)} \to \mathcal{W}^{(k)} \otimes \Omega^1_{M(\mathbb{C})}),$$
in particular $\mathcal{P}^{(k)}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_{M(\mathbb{C})} = \mathcal{W}^{(k)}$.

(iii) *$\mathcal{P}^{(k)}_{L,\Gamma_g}$ is the local system attached to the (monodromy) representation*
$$\pi_1(\Gamma_g \setminus X_+) = \Gamma_g \subset G^c(\mathbb{Q})_+ \to G^c(L) \to Gl(W).$$

(iv) *For all primes $\lambda \in L$ there exists a lisse L_λ-sheaf $\mathcal{P}^{(k)}_{\lambda,K,ét}$ (with L_λ being the λ-adic completion of L) on the étale side $(M_{K,\mathbb{C}})_{ét}$ such that*
$$(\mathcal{P}^{(k)}_{\lambda,K,ét})^{an} = \mathcal{P}^{(k)}_{L,K} \otimes_L L_\lambda.$$

Proof. (i) follows because Γ_g acts freely on X_+ (see Remark 3). For (ii) and (iii) just unravel the definitions. Finally the representation in (iii) is continuous in the λ-adic topology on the right and the profinite topology on the left. Hence we can complete both sides in the respective topology to obtain a representation
$$\pi_1(\Gamma_g \setminus X_+)_{\text{ét}} = \pi_1(\widehat{\Gamma_g \setminus X_+}) \to Gl(W \otimes_L L_\lambda)$$
and this gives the desired lisse sheaf (see SGA 5 VI 1). □

Theorem 9. *The lisse sheaves $\mathcal{P}^{(k)}_{\lambda,K}$ live already on $(M_K)_{ét}$ for all λ and $\mathcal{W}^{(k)}$, $F^{\alpha+p}$ and in particular $\mathcal{V}^{(k)}$ live already on $M_{\operatorname{Spec} L}$.*

Proof. For the second part see [Mi90] . For the first part assume λ is a prime over l. Then $\mathcal{P}^{(k)}_{\lambda,K}$ being a lisse sheaf means for all n there is a projective system of locally constant $\mathbb{Z}/l^n$-sheaves $\mathcal{P}^{(k)}_{n,K}$ on $(M_{K,\mathbb{C}})_{\text{ét}}$ such that $\mathcal{P}^{(k)}_{\lambda,K} = \varprojlim_n \mathcal{P}^{(k)}_{n,K} \otimes_{\mathbb{Z}_l} L_\lambda$. Since $\mathcal{P}^{(k)}_{n,K}$ is locally constant it is already defined over $(M_K)_{\text{ét}}$. □

Proposition 10. *Denote by $\mathcal{E}_K$ the sheaf of $\mathbb{C}$-valued C^∞-functions on $M_K(\mathbb{C})$. A subscript $(-)_{\mathcal{E}_K}$ will mean $\otimes_{\mathcal{O}_{M_K(\mathbb{C})}} \mathcal{E}_K$. Then*

- $\nabla F^p_K \subset F^{p-1}_K \otimes \Omega^1_{M_K(\mathbb{C})}$.
- $\mathcal{W}^{(k)}_{K,\mathcal{E}} = \bigoplus_{p+q=w-2} (\mathcal{W}^{(k)}_K)^{p,q}$*, with* $(\mathcal{W}^{(k)}_K)^{p,q} := F^p_{K,\mathcal{E}_K} \cap \overline{F^q_{K,\mathcal{E}_K}} \cong F^p_{\mathcal{E}_K}/F^{p+1}_{\mathcal{E}_K}$. *In particular* $(\mathcal{W}^{(k)}_K)^{p,q} = 0$ *if* $p \neq [\alpha, \alpha + k_1 - 2]$.

Proof. It is enough to prove the statement on $\mathbb{P}^1(\mathbb{C})$ and since the filtration only affects the first factor in

$$\mathcal{W}_{\mathbb{P}^1} = \mathrm{Sym}^{k_1-2}_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1} \otimes V_{\mathbb{C}}^{\vee}) \otimes W_{2,\mathbb{C}} \otimes \ldots \otimes W_{n,\mathbb{C}},$$

it is enough to prove the corresponding statement for $\mathcal{W}' := \mathrm{Sym}^{k_1-2}_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1} \otimes V_{\mathbb{C}}^{\vee})$. Then by (14)

$$F^{p+q}(\mathcal{W}') = (e_0 - ze_1)^p \cdot \mathrm{Sym}^{k_1-2-p}(\mathcal{O}_{\mathbb{P}^1} \otimes V_{\mathbb{C}}^{\vee}).$$

Now take $f = (e_0 - ze_1)^p v \in F^{\alpha+p}(\mathcal{W}')$. Then

$$\nabla f = -p(e_0 - ze_1)^{p-1} e_1 v \otimes dz + (e_0 - ze_1)^p \nabla v \in F^{\alpha+p-1}(\mathcal{W}') \otimes \Omega^1_{\mathbb{P}^1}. \tag{16}$$

This gives (i). For (ii) notice that $(e_0 - ze_1)$ and $(e_0 - \bar{z}e_1)$ is a basis for $\mathcal{E}_{\mathbb{P}^1} \otimes V_{\mathbb{C}}^{\vee}$. Thus

$$(e_0 - ze_1)^r (e_0 - \bar{z}e_1)^s, \quad \text{with } r \geq p \text{ and } r + s = k_1 - 2 \tag{17}$$

is a basis for $F^{\alpha+p}(\mathcal{W}')$. Hence for $\alpha + p + \alpha + q = w - 2$, i.e. $p + q = k_1 - 2$ we have

$$F^{\alpha+p}_{\mathcal{E}_{\mathbb{P}^1}} \cap \overline{F^{\alpha+q}_{\mathcal{E}_{\mathbb{P}^1}}} = (e_0 - ze_1)^p (e_0 - \bar{z}e_1)^q \mathcal{E}_{\mathbb{P}^1}.$$

This gives

$$\mathcal{W}' = \oplus_{\alpha+p+\alpha+q=w-2} F^{\alpha+p}_{\mathcal{E}_{\mathbb{P}^1}} \cap \overline{F^{\alpha+q}_{\mathcal{E}_{\mathbb{P}^1}}}.$$

The isomorphism $F^{\alpha+p}_{\mathcal{E}_{\mathbb{P}^1}} \cap \overline{F^{\alpha+q}_{\mathcal{E}_{\mathbb{P}^1}}} \cong F^{\alpha+p}_{\mathcal{E}_{\mathbb{P}^1}} / F^{\alpha+p+1}_{\mathcal{E}_{\mathbb{P}^1}}$ is clear. □

We still have $G^c(\mathbb{C})$ acting on $\mathcal{W}^{(k)}_{\Gamma_g}$ for all Γ_g. An element

$$\gamma = (\gamma_1, \ldots, \gamma_n) \in Gl(\mathbb{R})_+ = Gl_2(\mathbb{R})_+ \times \mathbb{H}^\times \times \ldots \times \mathbb{H}^\times$$

defines a point $P_\gamma \in X_+$. If we identify X_+ with the upper half plane, then

$$P_\gamma = \gamma_1.i = \frac{ai+b}{ci+d}, \quad \gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In this case denote by $\bar{P}_\gamma$ the image of P_γ in $\Gamma_g \setminus X_+$. If we identify X_+ with $\{gh(-)g^{-1} \,|\, g \in G(\mathbb{R})_+\}$, then

$$P_\gamma = h_\gamma := \gamma h(-)\gamma^{-1} : \mathbb{S}(\mathbb{R}) \to G(\mathbb{R}).$$

We obtain an action of $\mathbb{S}(\mathbb{R})$ on the fiber $\mathcal{W}^{(k)}_{\Gamma_g,(\bar{P}_\gamma)} = \mathcal{W}^{(k)}_{\Gamma_g} \otimes_{\mathcal{O}_{\bar{P}_\gamma}} k(\bar{P}_\gamma)$ via

$$\xi_\gamma := R^{(k)}_{(\bar{P}_\gamma)} \circ \mathrm{nat} \circ h_\gamma : \mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \xrightarrow{h_\gamma} G(\mathbb{R}) \xrightarrow{\mathrm{nat}} G^c(\mathbb{C}) \xrightarrow{R^{(k)}_{(\bar{P}_\gamma)}} \mathrm{Aut}(\mathcal{W}^{(k)}_{\Gamma,(\bar{P}_\gamma)}), \tag{18}$$

where nat denotes the natural map.

Proposition 11. *Denote*

$$W^{p,q}_\gamma := \{w \in \mathcal{W}_{\Gamma_g,(\bar{P}_\gamma)} \,|\, \xi_\gamma(z)(w) = \frac{1}{z^p}\frac{1}{\bar{z}^q} w, \forall z \in \mathbb{C}^\times\}.$$

Then

$$W^{p,q}_\gamma = \begin{cases} F^p_{\Gamma_g,(\bar{P}_\gamma)} / F^{p+1}_{\Gamma_g,(\bar{P}_\gamma)} & \text{if } p+q = w-2 \\ 0 & \text{else.} \end{cases}$$

In particular $W^{p,q}_\gamma = 0$ *if* $p \notin [\alpha, \alpha + k_1 - 2]$ *and* $W^{\alpha+k_1-2,\alpha}_\gamma \cong \mathcal{V}^{(k)}_{\Gamma_g,(\bar{P}_\gamma)}$.

Proof. It is enough to prove the corresponding statement on $\mathbb{P}^1(\mathbb{C})$. We have

$$\mathcal{W}^{(k)}_{\mathbb{P}^1(\mathbb{C}),P} \cong W_{\mathbb{C}} = W_{1,\mathbb{C}} \otimes \ldots \otimes W_{n,\mathbb{C}}.$$

By definition of h, $\mathbb{S}(\mathbb{R})$ acts non-trivially only on $W_{1,\mathbb{C}} = \mathrm{Sym}^{k_1-2}(V_{\mathbb{C}}^{\vee})$. Thus it is enough to prove the corresponding statement for $\mathcal{W}' := \mathrm{Sym}^{k_1-2}(\mathcal{E}_{\mathbb{P}^1} \otimes V_{\mathbb{C}}^{\vee})$ and to assume $\gamma = \gamma_1$. Set

$$e_{0,\gamma} := (e_0 - ie_1)\gamma^{-1}, \quad e_{1,\gamma} = (e_0 + ie_1)\gamma^{-1}.$$

Then for $z = x + iy$

$$e_{0,\gamma}.(\gamma h(z)\gamma^{-1}) = (e_0 - ie_1)\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = ze_{0,\gamma}$$

and

$$e_{1,\gamma}(\gamma h(z)\gamma^{-1}) = \bar{z}e_{1,\gamma}.$$

Thus for $r + s = k_1 - 2$

$$\xi(z)(e_{0,\gamma}^r e_{1,\gamma}^s) = (\det(h_\gamma(z))^{-1})^\alpha (e_{0,\gamma}h_\gamma(z)^{-1})^r (e_{1,\gamma}h_\gamma(z)^{-1})^s \tag{19}$$

$$= \frac{1}{|z|^{2\alpha}}\frac{1}{z^r\bar{z}^s}e_{0,\gamma}^r e_{1,\gamma}^s \tag{20}$$

$$= \frac{1}{z^{r+\alpha}}\frac{1}{\bar{z}^{s+\alpha}}e_{0,\gamma}^r e_{1,\gamma}^s. \tag{21}$$

This shows $W_\gamma^{p,q}$ is spanned by $e_{0,\gamma}^{p-\alpha}e_{1,\gamma}^{q-\alpha}$ if $p + q = w - 2$ and $p \in [\alpha, \alpha + k_1 - 2]$ and is 0 else. On the other hand $F_{\mathcal{E}}^p$ is spanned by (see (17))

$$(e_0 - ze_1)^r(e_0 - \bar{z}e_1)^s, \quad \text{with } r \geq p \text{ and } r + s = k_1 - 2.$$

Now the statement follows since in the fiber P_γ the vectors

$$(e_0 - ze_1)_{(P_\gamma)} = (e_0 - (\gamma.i)e_1) \quad \text{and} \quad (e_0 - \bar{z}e_1)_{(P_\gamma)} = (e_0 + (\gamma.i)e_1)$$

are multiples of $e_{0,\gamma}$ and $e_{1,\gamma}$ respectively, by (9). □

Recall the following definition.

Definition 12. Let $\mathcal{P}$ be a local system of $\mathbb{R}$-vector spaces on a smooth projective manifold S. A *variation of Hodge structures (VHS) of weight* k on $\mathcal{P}$ is a filtration

$$\mathcal{P} \otimes_{\mathbb{R}} \mathcal{O}_S \supset \ldots \supset F^p \supset F^{p-1} \supset \ldots,$$

such that

(i) $\nabla F^p \subset F^{p-1} \otimes \Omega^1_S$, where $\nabla : \mathcal{P} \otimes \mathcal{O}_S \to \mathcal{P} \otimes \Omega^1_S$ is the connection defined by the local system.

(ii) $\mathcal{P} \otimes_{\mathbb{R}} \mathcal{E}_S = \oplus_{p+q=k}\mathcal{E}^{p,q}(\mathcal{P})$, with $\mathcal{E}^{p,q}(\mathcal{P}) := F_{\mathcal{E}}^p \cap \overline{F_{\mathcal{E}}^q}$, $p + q = k$.

A VHS of weight k is thus a Hodge structure of weight k in each fiber $\mathcal{P}_{(s)}$, $s \in S$, varying holomorphically.

There is also the notion of a polarization of a VHS on $\mathcal{P}$, which we omit.

Theorem 13 (Deligne, see [Zu79] Thm (2.9))**.** *Let $\mathcal{P}$ be a VHS of weight k as above, which admits a polarization. Then $H^i(S, \mathcal{P}_{\mathbb{C}})$ is a Hodge structure of weight $k + i$ and there is a* canonical *decomposition*

$$H(S, \mathcal{P}_{\mathbb{C}}) = \oplus_{p+q=i}H^{p+q}(S, Gr_F^p\Omega^{\cdot}_S(\mathcal{P})).$$

Remark 14. Proposition 10 shows that $\mathcal{P}^{(k)}_{K,\mathbb{C}}$ is almost a VHS. The only missing thing is that $\mathcal{P}^{(k)}_{K,\mathbb{C}}$ has no underlying local system of $\mathbb{R}$-vector spaces, since L is not real. We can fix this problem by simply considering $\mathcal{P}^{(k)}_{K,\mathbb{C}}$ as $\mathbb{R}$-vector spaces, i.e. let $\sigma : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ be the natural map and consider $\sigma_*\mathcal{P}^{(k)}_{K,\mathbb{C}}$. Then

$$\sigma_*\mathcal{P}^{(k)}_{K,\mathbb{C}} \otimes_{\mathbb{R}} \mathcal{O}_{M_K(\mathbb{C})} \cong \mathcal{W}^{(k)}_{K,\mathbb{C}} \oplus \overline{\mathcal{W}^{(k)}_{K,\mathbb{C}}} = \mathcal{W}^{(k)}_{K,\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C},$$

where we view $\mathcal{W}^{(k)}_{K,\mathbb{C}}$ and $\overline{\mathcal{W}^{(k)}_{K,\mathbb{C}}} \subset \mathcal{P}^{(k)}_{K,\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{E}_{M_K(\mathbb{C})}$. This decomposition is induced by

$$\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}_{M_K(\mathbb{C})} \cong \mathcal{O}_{M_K(\mathbb{C})} \oplus \overline{\mathcal{O}_{M_K(\mathbb{C})}} \subset \mathcal{E}_{M_K(\mathbb{C})}, \quad (1+i)\otimes f + (1-i)\otimes g \mapsto (f, \bar{g}).$$

We obtain a filtration

$$\mathrm{Fil}^p_K := F^p_K \otimes_{\mathbb{R}} \mathbb{C} = F^p_K \oplus \overline{F^p_K} \text{ on } \sigma_*\mathcal{P}^{(k)}_{K,\mathbb{C}} \otimes_{\mathbb{R}} \mathcal{O}_{M_K(\mathbb{C})},$$

which satisfies the analog statements of the Propositions 10 and 11. Thus $\sigma_*\mathcal{P}^{(k)}_{K,\mathbb{C}}$ is a VHS of weight $w-2$. It follows from Proposition 11, that this VHS coincides with the one coming from the machinery of Shimura varieties (see [De79], [Mi90]). We use as a black box that there exists also a polarization of this VHS (see [De79], [Mi90]). (Probably one can describe this very concretely as in [BaNe81].)

Theorem 15 ([Sa06], Proof of Lemma 1)**.** *There is a canonical isomorphism of $\mathbb{C}[G(\mathbb{A}_f)]$-modules*

$$\begin{aligned} H^1(M(\mathbb{C})), \mathcal{P}^{(k)}) &\cong QM^{(k)} \oplus \overline{QM^{(k)}} \\ &\cong H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1_{M(\mathbb{C})}) \oplus \overline{H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1_{M(\mathbb{C})})}. \end{aligned}$$

Proof. One easily checks $QM^{(k)} = H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1)$. Further by Theorem 13 and Remark 14 we have a canonical decomposition (in particular compatible with the $G(\mathbb{A}_f)$-action)

$$H^1(M(\mathbb{C}), \sigma_*\mathcal{P}^{(k)} \otimes_{\mathbb{R}} \mathbb{C}) \cong \oplus_{p+q=1} H^{p+q}(M(\mathbb{C}), \mathrm{Gr}^p_{\mathrm{Fil}}\Omega^{\cdot}(\sigma_*\mathcal{P}^{(k)})).$$

On the other hand

$$H^1(M(\mathbb{C}), \sigma_*\mathcal{P}^{(k)} \otimes_{\mathbb{R}} \mathbb{C}) \cong H^1(M(\mathbb{C}), \mathcal{P}^{(k)}) \oplus \overline{H^1(M(\mathbb{C}), \mathcal{P}^{(k)})}$$

and

$$\begin{aligned} &H^{p+q}(M(\mathbb{C}), \mathrm{Gr}^p_{\mathrm{Fil}}\Omega^{\cdot}(\sigma_*\mathcal{P}^{(k)})) \\ &\qquad \cong H^{p+q}(M(\mathbb{C}), \mathrm{Gr}^p_F\Omega^{\cdot}(\mathcal{P}^{(k)})) \oplus \overline{H^{p+q}(M(\mathbb{C}), \mathrm{Gr}^p_F\Omega^{\cdot}(\mathcal{P}^{(k)}))}. \end{aligned}$$

The Theorem now follows from the following Lemma: □

Lemma 16.

$$H^1(M(\mathbb{C}), Gr^p_F\Omega^{\cdot}(\mathcal{P}^{(k)})) \cong \begin{cases} H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1) & \text{if } p \in \{\alpha, \alpha + k_1 - 1\} \\ 0 & \text{else.} \end{cases}$$

Proof. The filtration on $\Omega^{\cdot}(\mathcal{P}^{(k)})$ looks as follows:

$$
\begin{array}{ccccc}
\Omega^{\cdot}(\mathcal{P}^{(k)}): & & \mathcal{W}^{(k)} & \xrightarrow{\nabla} & \mathcal{W}^{(k)}\otimes_{\mathcal{O}}\Omega^1 \\
\| & & \| & & \| \\
F^{\alpha}\Omega^{\cdot}(\mathcal{P}^{(k)}): & & F^{\alpha} & \xrightarrow{\nabla} & F^{\alpha-1}\otimes_{\mathcal{O}}\Omega^1 \\
\uparrow & & \uparrow & & \| \\
F^{\alpha+1}\Omega^{\cdot}(\mathcal{P}^{(k)}): & & F^{\alpha+1} & \xrightarrow{\nabla} & F^{\alpha}\otimes_{\mathcal{O}}\Omega^1 \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
F^{\alpha+k_1-2}\Omega^{\cdot}(\mathcal{P}^{(k)}): & & F^{\alpha+k_1-2} & \xrightarrow{\nabla} & F^{\alpha+k_1-3}\otimes_{\mathcal{O}}\Omega^1 \\
\uparrow & & \uparrow & & \uparrow \\
F^{\alpha+k_1-1}\Omega^{\cdot}(\mathcal{P}^{(k)}): & & 0 & \xrightarrow{\nabla} & F^{\alpha+k_1-2}\otimes_{\mathcal{O}}\Omega^1 \\
\uparrow & & \uparrow & & \uparrow \\
F^{\alpha+k_1}\Omega^{\cdot}(\mathcal{P}^{(k)}): & & 0 & & 0.
\end{array}
$$

It follows that $\mathrm{Gr}_F^p(\Omega^{\cdot}(\mathcal{P}^{(k)})) = 0$ for $p < \alpha$ and $p > \alpha + k_1 - 1$. Now assume $\alpha < p < \alpha + k_1 - 1$. Then

$$\mathrm{Gr}_F^p\nabla : F^p/F^{p+1} \to F^{p-1}/F^p \otimes \Omega^1$$

is an isomorphism. Indeed it is enough to check this on $\mathbb{P}^1(\mathbb{C})$ and then it follows from (16) that $\mathrm{Gr}_f^p(\nabla)$ is given by $(e_0 - ze_1)^p v \mapsto -p(e_0 - ze_1^{p-1})e_1 v \otimes dz \bmod F^p$, which is an isomorphism. Thus

$$H^1(M(\mathbb{C}), \mathrm{Gr}_F^p(\Omega^{\cdot}(\mathcal{P}^{(k)}))) = 0 \quad \text{for } \alpha < p < \alpha + k_1 - 1.$$

If $p = \alpha + k_1 - 1$, then

$$\mathrm{Gr}_F^p(\Omega^{\cdot}(\mathcal{P}^{(k)})) = (0 \to \mathcal{V}^{(k)} \otimes \Omega^1)$$

and hence

$$H^1(M(\mathbb{C}), \mathrm{Gr}_F^{\alpha+k_1-1}(\Omega^{\cdot}(\mathcal{P}^{(k)}))) = H^0(M(\mathbb{C}, \mathcal{V}^{(k)}) \otimes \Omega^1).$$

Finally for $p = \alpha$ we have

$$\mathrm{Gr}_F^{\alpha}(\Omega^{\cdot}(\mathcal{P}^{(k)})) = (F^{\alpha}/F^{\alpha+1} \to 0)$$

and it follows from (15) that $F^{\alpha}/F^{\alpha+1} \cong (\mathcal{V}^{(k)})^{\vee}$. Thus

$$H^1(M(\mathbb{C}), \mathrm{Gr}_F^{\alpha}(\Omega^{\cdot}(\mathcal{P}^{(k)}))) \cong H^1(M(\mathbb{C}), \mathcal{V}^{(k)\vee}) \cong H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1)$$

the last isomorphism being Serre duality, which is also compatible with the $G(\mathbb{A}_f)$-action. This proves the Lemma and hence the Theorem.

□

Remark 17. Going attentively through the proof one sees that we obtain in fact

$$H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1_{M(\mathbb{C})}) \cong \overline{H^0(M(\mathbb{C}), \mathcal{V}^{(k)} \otimes \Omega^1_{M(\mathbb{C})})}$$

and one can check that the isomorphism from Theorem 15 is induced by tensoring the following isomorphism with $\otimes_{L_\lambda}\mathbb{C}$

$$H^1_{\text{ét}}(M_L, \mathcal{P}_\lambda^{(k)}) \cong (H^0(M_L, \mathcal{V}_L^{(k)} \otimes \Omega^1_{M_L}) \otimes_L L_\lambda)^{\oplus 2},$$

where λ is any prime in L and $\mathcal{P}_\lambda^{(k)}$ and $\mathcal{V}_L^{(k)}$ are the sheaves given by Theorem 9, see [Sa06, Proof of Lemma 1] for details.

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