# Quaternion Algebras over Global Fields

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**Abstract**

This is an expository work of the classification of Quaternion Algebras over algebraic number fields for the Forschungsseminar Sommersemester 08

## 1 Introduction

In this section I'll recall some results of the previous talks. Let $K$ be a field with $\operatorname{char} K \neq 2$ and let $H$ be a quaternion algebra over $K$. We can find a basis of $H$ of the form $\{1, i, j, ij\}$ such that $i^2 = a$, $j^2 = b$, $ij = -ji$ for some $a, b \in K^*$. We denote then $H = \left(\frac{a,b}{K}\right)$.

**Example 1.1**
Let $K$ be a field. The $K$-algebra $\left(\frac{1,-1}{K}\right)$ is isomorphic to the matrix algebra $\mathrm{Mat}_2(K)$.

Let $(\overline{\ })$ denotes the involution of $H$ such that if $x = x_0 + x_1 i + x_2 j + x_3 ij$, then $\overline{x} = x_0 - x_1 i - x_2 j - x_3 ij$. Define

$$\begin{aligned} N(x) &= x \cdot \overline{x} \quad \textit{reduced norm} \\ Tr(x) &= x + \overline{x} \quad \textit{reduced trace} \end{aligned}$$

as functions from $H$ into $K$, for all $x \in H$. Furthermore, the reduced norm is a quadratic form of rank 4 over $K$ with corresponding bilinear form $B(x, y) = Tr(x\overline{y})$. The set of *pure* quaternion elements of a quaternion algebra $H = \left(\frac{a,b}{K}\right)$ is defined as

$$H_0 = \{x \in H | Tr(x) = 0\}.$$

The following result gives us a characterization of $H$ in terms of quadratic forms:

***Proposition 1.1***
*Let $H$, $H'$ be quaternion algebras over a field $K$. Then the following are equivalent:*

1. *$H$ and $H'$ are algebra isomorphic.*
2. *$(H, N)$ and $(H', N')$ are isometric quadratic spaces.*
3. *$(H_0, N|_{H_0})$ and $(H'_0, N|_{H'_0})$ are isometric quadratic spaces.*

We then turn to world of quadratic spaces in order to obtain some classifications in the case when the field $K$ is a global field.

## 2 Basics on Quadratic Forms

Let $(V, Q)$ be a quadratic space and a basis $\{v_i, \ldots, v_n\}$. The *discriminant* of $V$ is defined as

$$dV := \det[B(v_i, v_j)] \mod K^{*2}.$$

If $W$ is a subspace of $V$, we say that $W$ *splits* $V$ if there exist $U$ subspace of $V$ such that $V \cong W \perp U$ (here $B(W, U) = 0$). It is clear that if $V \cong V_1 \perp \cdots \perp V_r$, then $dv = dV_1 \cdots dV_r$.
Let $a_1, \ldots, a_n \in K^*$. The notation $V \cong \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ means that we have an orthogonal basis $\{v_1, \ldots, v_n\}$ of $V$ and $Q(v_i) = a_i$ for all $i$. In this case $dV = a_1 \cdots a_n$.

**Example 2.1**
Let $H = \left(\frac{a,b}{K}\right)$ be a quaternion algebra. Then $H \cong \langle 1 \rangle \perp \langle a \rangle \perp \langle b \rangle \perp \langle ab \rangle$ and hence $dH = 1$.

***Definition 2.1***
*Let $(V, Q)$ be a quadratic space.*

1. *The space $(V, Q)$ is called* isotropic *if $Q$ represents 0 (i.e. exist a non-zero vector $v \in V$ s.t. $Q(v) = 0$).*

2. *The space $(V, Q)$ is called* regular *if*

$$\{v \in V | B(v, V) = 0\} = \{\overline{0}\}$$

*(in our case this is equivalent to say that $B$ is non-degenerate)*

**Remark 1**
Let $(V, Q)$ be a regular quadratic space over a field $K$, $a \in K^*$. Then $a \in Q(V) \iff \langle -a \rangle \perp V$ is isotropic.

We finish this section by giving an elementary but useful lemma in order to characterize when a quaternion algebra is a matrix algebra.

***Lemma 1***
*let $K$ be a field, $a, b \in K^*$. Then the following are equivalent.*

1. $\left(\frac{a,b}{K}\right) \cong \left(\frac{1,-1}{K}\right)$.

2. $\left(\frac{a,b}{K}\right)$ *is not a division algebra.*

3. $(\left(\frac{a,b}{K}\right), N)$ *is isotropic.*

4. $(\left(\frac{a,b}{K}\right)_0, N')$ *is isotropic (here $N'$ is $N$ restricted to $\left(\frac{a,b}{K}\right)_0$).*

5. $\langle a \rangle \perp \langle b \rangle$ *represents 1.*

6. $a \in N_{F/K}(F)$ *where $F = K[\sqrt{b}]$.*

# 3 Quadratic Forms over Global Fields

Let now $K$ be a global field; for instance we suppose that $K$ is an algebraic number field and everthing is equivalent in the function field case. Denote $\Omega_K$ the set of (non-trivial) places of $K$. A quadratic $K$-space $(V, Q)$ is called *isotropic at* $\mathfrak{p} \in \Omega_K$ if the space $V_{\mathfrak{p}} := (V \otimes_K K_{\mathfrak{p}}, Q)$ is isotropic.

We have then the first important result called *Hasse Principle* related to quadratic spaces:

***Theorem 3.1***
*A regular quadratic space over a global field $K$ is isotropic if and only if it is isotropic at all $\mathfrak{p} \in \Omega_K$.*

and as a direct consecuence:

***Theorem 3.2 (Hasse-Minkowski)***
*Let $U, V$ regular quadratic spaces over a global field $K$. Then $U$ is isometric to $V$ if and only if $U_{\mathfrak{p}}$ is isometric to $V_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega_K$.*

***Definition 3.1***
*Let $H$ be a quaternion algebra over a global field $K$, and $\mathfrak{p} \in \Omega_K$. We say that $H$ is* ramified at $\mathfrak{p}$ *if $H_{\mathfrak{p}} \cong \mathbb{H}_{\mathfrak{p}}$ where $\mathbb{H}_{\mathfrak{p}}$ denotes the division algebra over $K_{\mathfrak{p}}$. If $\mathfrak{p}$ is a real place one uses* definite/indefinite *instead of* ramified/unramified.

Then we have our first classification theorem of quaternion algebras over global fields:

***Theorem 3.3***
*Let $K$ be an algebraic number field, $H$ and $H'$ quaternion algebras over $K$. Then the following are equivalent.*

1. *$H \cong H'$.*
2. *$H_{\mathfrak{p}} \cong H'_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega_K$.*
3. *$H$ and $H'$ ramifies at the same places.*

***Definition 3.2***
*Let $K$ be a local field. given $a, b \in K^*$, we define their* Hilbert symbol *:*

$$(a, b) = \begin{cases} +1 & \text{if } X^2 - aY^2 - bZ^2 \text{ represents } 0 \\ -1 & \text{otherwise} \end{cases}$$

**Remark 2**
Let $H = \left(\frac{a,b}{K}\right)$ defined over a global field and $\mathfrak{p} \in \Omega_K$. By lemma 1

$$\begin{aligned} H \text{ is unramified at } \mathfrak{p} &\iff H_{\mathfrak{p}} \text{ is not a division algebra} \\ &\iff X^2 - aY^2 - bZ^2 \text{ represents } 0 \\ &\iff +1 = (a, b)_{\mathfrak{p}} := (a, b) \text{ in } K_{\mathfrak{p}}. \end{aligned}$$

***Proposition 3.1 (Properties of the Hilbert Symbol)***
*Let $a, b, c \in K^*$, $\mathfrak{p} \in \Omega_K$.*

*1.* *(a)* $(a,bc)_\mathfrak{p} = (a,b)_\mathfrak{p}(a,c)_\mathfrak{p}$

*(b)* $(a,-a)_\mathfrak{p} = 1$

*(c)* $(a,b^2)_\mathfrak{p} = 1$.

*2. If* $\mathfrak{p}$ *is a real place* $(a,c)_\mathfrak{p} = -1$ *if* $a < 0$ *and* $b < 0$.

*3.* *(a)* $(a,b)_\mathfrak{p} = 1$ *if* $a, b \in R_\mathfrak{p}^*$.

*(b)* $(a,\mathfrak{p})_\mathfrak{p} = \left(\frac{a}{\mathfrak{p}}\right)$ *if* $a \in R_\mathfrak{p}^*$.

*4.* $(a,b)_\mathfrak{p} = 1$ *for almost all* $\mathfrak{p} \in \Omega_K$ *and*

$$\prod_{\mathfrak{p}\in\Omega_K} (a,b)_\mathfrak{p} = 1. \qquad \text{(The Product Formula)}$$

*Proof.* see for example [Milne, lemma 6.6] or [O'Meara, ♮ 63,71].

***Corollary 3.1***
*Let $H$ be a quaternion algebra over an algebraic number field $K$. Then $H$ is ramified at an even number of places of $K$.*

***Definition 3.3***
*Let $H$ be a quaternion algebra over an algebraic number field $K$. Define the* (reduced) discriminant of $H$ *as*

$$DH := \mathfrak{p}_1 \cdots \mathfrak{p}_r$$

*where the $\mathfrak{p}_i'$s are exactly the places at which $H$ ramifies.*

There is a natural question: If we have $\mathcal{A}_\mathfrak{p}$ quaternion algebras for each $\mathfrak{p} \in \Omega_K$, is it posible to find $H$ a quaternion algebra over $K$ such that $H_\mathfrak{p} \cong \mathcal{A}_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega_K$?

***Definition 3.4***
*Let $(V,Q)$ a regular quadratic space over a global field $K$ and a representation of $V$ as $V \cong \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$. We define the* Hasse Invariant of $V$ with respect to $\mathfrak{p} \in \Omega_K$ *as*

$$S_\mathfrak{p} V = \prod_{1 \le i \le j \le n} (a_i, a_j)_\mathfrak{p}.$$

**Remark 3**
The Hasse invariant of a quadratic space $V$ over a local field $K$ only depends of the isometry class of $V$.

*Proof.* see [Milne, prop 6.7].
We can state the principal result of this section which gives us an answer to the above question:

***Theorem 3.4***
*Let $K$ be a global field and, for each $\mathfrak{p} \in \Omega_K$, suppose given $U_\mathfrak{p}$ regular quadratic spaces over $K_\mathfrak{p}$ with $\dim U_\mathfrak{p} = n$. Then there exist $V$ regular quadratic space over $K$ with $\dim V = n$ such that $V_\mathfrak{p} \cong U_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega_K$ if and only if the following holds*

1. *There exists* $d_0 \in K^*$ *such that* $dU_{\mathfrak{p}} = d_0$ *for all* $\mathfrak{p} \in \Omega_K$.

2. *For almost all* $\mathfrak{p} \in \Omega_K$ *we have* $S_{\mathfrak{p}}U_{\mathfrak{p}} = 1$.

3. $\prod_{\mathfrak{p}\in\Omega_K} S_{\mathfrak{p}}U_{\mathfrak{p}} = 1$.

*proof:*

($\Rightarrow$) direct from the product formula and $d_0 = dV$.
($\Leftarrow$) We begin with a remark that will be useful for the rest of the proof.

**Remark 4**
If $U, V$ are regular quadratic spaces over a local field, then they are isometric if and only if

$$\dim U = \dim V \qquad dU = dV \qquad SU = SV$$

*Proof.* (of the remark) see [O'Meara, 63:20].
if $n = 1$ the result follows taking $V \cong \langle d_0 \rangle$. Assume then $n \geq 2$.
Let $T$ the subset of $\Omega_K$ which contains all the archimedean places of $K$ and all the finite places $\mathfrak{p}$ such that $S_{\mathfrak{p}}U_{\mathfrak{p}} = -1$. For each $\mathfrak{p} \in T$ we can write

$$U_{\mathfrak{p}} \cong \langle a_{1,\mathfrak{p}} \rangle \perp \cdots \perp \langle a_{n,\mathfrak{p}} \rangle$$

where $a_{i,\mathfrak{p}} \in K_{\mathfrak{p}}^*$, for all $1 \leq i \leq n$ and for all $\mathfrak{p} \in T$. Using the weak approximation theorem we can find $a_i \in K^*$ such that $|a_{i,\mathfrak{p}} - a_i|_{\mathfrak{p}}$ is small enough. Since $K_{\mathfrak{p}}^{*2}$ is an open subset of $K_{\mathfrak{p}}$ we can furthermore get $a_i \in a_{i,\mathfrak{p}}K_{\mathfrak{p}}^{*2}$ for all $\mathfrak{p} \in T$. Doing this for all $1 \leq i \leq n-1$ consider the quadratic space $W$ such that

$$W \cong \langle a_1 \rangle \perp \cdots \perp \langle a_{n-1} \rangle \perp \langle a_1 \cdots a_{n-1} d_0 \rangle.$$

By this choice we have that $W_{\mathfrak{p}} \cong U_{\mathfrak{p}}$ for all $\mathfrak{p} \in T$, but this occurs in a bigger subset of $\Omega_K$; for instance, if we define $R = \{\mathfrak{p} \in \Omega_K | S_{\mathfrak{p}}W_{\mathfrak{p}} \neq S_{\mathfrak{p}}U_{\mathfrak{p}}\}$, then we have $W_{\mathfrak{p}} \cong U_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega_K \setminus R$. If $R = \emptyset$ we are done. If not, $R$ is a finite subset of $\Omega_K$ having an even number of elements. We can also see that $R = \{\mathfrak{p} \in \Omega_K | S_{\mathfrak{p}}W_{\mathfrak{p}} = -1\}$.

**Claim:** There exist $P, P'$ quadratic planes such that $dP = dP'$ and $P_{\mathfrak{p}} \cong P'_{\mathfrak{p}} \iff \mathfrak{p} \in R$.

for this we need the following lemma:

***Lemma 2***
*Let* $K$ *an algebraic number field,* $b \in K^*$ *and* $T \subset \Omega_K$ *finite. If* $T$ *has an even number of elements and* $b$ *does not become an square in* $K_{\mathfrak{p}}$ *for* $\mathfrak{p} \in T$*, then there exist* $a \in K^*$ *such that*

$$(a,b)_{\mathfrak{p}} = \begin{cases} +1 & \mathfrak{p} \notin T \\ -1 & \textit{otherwise} \end{cases}$$

*Proof.* For a proof of this Lemma see [O'Meara, 71:19] or [Milne, 6.13].
Applying this lemma for $R$ and any $b \in K^*$ such that it is a non-square at $\mathfrak{p} \in R$, exist $a \in K^*$ such that

$$(a,b)_{\mathfrak{p}} = \begin{cases} +1 & \mathfrak{p} \notin R \\ -1 & \text{otherwise} \end{cases}$$

Take now

$$\begin{aligned} P &\cong \langle 1 \rangle \perp \langle -b \rangle \quad \text{and} \\ P' &\cong \langle a \rangle \perp \langle -ab \rangle. \end{aligned}$$

These two planes have the same discriminant and dimension, and computing their Hasse invariant we get

$$S_{\mathfrak{p}} P_{\mathfrak{p}} = (-1,b)_{\mathfrak{p}} \qquad S_{\mathfrak{p}} P'_{\mathfrak{p}} = (-1,b)_{\mathfrak{p}} (a,b)_{\mathfrak{p}},$$

hence $P_{\mathfrak{p}} \cong P'_{\mathfrak{p}} \iff \mathfrak{p} \notin R$, and we have proved the claim.

Consider $\mathfrak{p} \in R$. We have that $P'_{\mathfrak{p}} \perp U_{\mathfrak{p}} \cong (P \perp W)_{\mathfrak{p}}$ because the three invariants coincide, so there exists a representation $P'_{\mathfrak{p}} \longrightarrow (P \perp W)_{\mathfrak{p}}$.
If $\mathfrak{p} \notin R$ we have that $P_{\mathfrak{p}} \cong P'_{\mathfrak{p}}$ and there exists also a representation $P'_{\mathfrak{p}} \longrightarrow (P \perp W)_{\mathfrak{p}}$.
The Hasse principle tells us that we get a global representation $P' \longrightarrow P \perp W$. Using Witt's Theorem there exist a regular quadratic space $V$ over $K$ such that

$$P' \perp V \cong P \perp W.$$

For the space $V$ holds the following:

1. $dV = dW = d_0$ because $dP = dP'$.
2. For each $\mathfrak{p} \notin R$ we have $P_{\mathfrak{p}} \cong P'_{\mathfrak{p}}$ and hence $V_{\mathfrak{p}} \cong W_{\mathfrak{p}} \cong U_{\mathfrak{p}}$.
3. If $\mathfrak{p} \in R$ obviously $V_{\mathfrak{p}} \ncong W_{\mathfrak{p}}$, hence $S_{\mathfrak{p}} V_{\mathfrak{p}} = -S_{\mathfrak{p}} W_{\mathfrak{p}} = S_{\mathfrak{p}} U_{\mathfrak{p}}$ and this implies also that $V_{\mathfrak{p}} \cong U_{\mathfrak{p}}$.

We have then proved the theorem. ∎

***Corollary 3.2***
*Given an even number of places of an algebraic number field it is always possible to find a quaternion algebra which is ramified at exactly these places.*

# References

[O'Meara] O'Meara, O.T., *Introduction to Quadratic Forms*, 117, Die Grundlehren der mathematischen Wissenshaften, Springer-Verlag, 1973.

[Milne] Milne, J.S., *Class Field Theory (v4.00)*, Available at www.jmilne.org/math/, 2008.