SUPERSPECIAL POINTS ON HILBERT MODULAR VARIETIES AND THE ENDOMORPHISMS ORDERS, PRELIMINARY VERSION

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1. Introduction

These are companion notes to the talk given on April 17th in Essen in the Forschungsseminar on Quaternion Algebras, based mostly on the papers [17] and [18].

We describe in these notes quaternionic endomorphism orders of abelian varieties with real multiplication in characteristic $p > 0$, and discuss some illustrative examples.

Recall that a g-dimensional abelian variety A is said to have real multiplication (or RM for short) if it is equipped with the action of the ring of integers $\mathcal{O}_L$ of a totally real field L of dimension $[L : \mathbb{Q}] = g$. We start with some sketchy, partly historical remarks with the motivational goal of providing the geometric picture in dimension one before addressing a generalization to higher dimensions.

Let H be the class number of $B_{p,\infty}$ i.e., the number of left ideal classes of a maximal order in the rational quaternion algebra $B_{p,\infty}$ ramified at p and ∞. Let $I_i, I_j, 1 \leq i, j \leq H$ be left ideal classes representatives. Using the norm of the quaternion algebra, we can define:

$$Q_{ij}(x) := \mathrm{Norm}(x)/\mathrm{Norm}(I_j^{-1}I_i), \text{ for } x \in I_j^{-1}I_i,$$

i.e., a quadratic form of level p, discriminant p^2, with values in $\mathbb{N}$. Since the quaternion algebra $B_{p,\infty}$ is definite (i.e., ramified at the infinite place), the representation numbers $a(n) := |\{x | Q_{ij}(x) = n\}|$ are finite. The theta series

$$\theta_{ij}(z) := \sum_{n \in \mathbb{N}} a(n) q^n, \qquad \text{for } q = e^{2\pi i z},$$

is a modular form of weight 2 for $\Gamma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) | \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod p\}$ by the Poisson summation formula. In 1954, Eichler ([5]) showed that the $H(H-1)$ cusp forms

$$\theta_{ij}(z) - \theta_{1j}(z), \quad 2 \leq i \leq H, 1 \leq j \leq H,$$

span the vector space $S_2(\Gamma_0(p))$ of cusp forms of weight 2 for the group $\Gamma_0(p)$. Hecke had originally conjectured in 1940 ([10, p. 884-885]) that $H - 1$ differences of theta series (say, obtained from fixing the index j in the above formulation) would form a basis of $S_2(\Gamma_0(p))$, maybe inspired by the similarity of the explicit formulae for the class number (of a maximal order) of $B_{p,\infty}$ and for the dimension of $S_2(\Gamma_0(p))$ (see below Remark 1.2). In spite of this striking coincidence, Hecke's conjecture holds only for $p \leq 31$, and $p = 41, 47, 59, 71$ (cf. [20, Rmk. 2.16]). For further historical remarks on the Basis Problem, we refer to [20] and the references therein.

We now introduce some geometric notions.

Definition 1.1. *An elliptic curve E over $\overline{\mathbb{F}}_p$ is* supersingular *if $E[p](\overline{\mathbb{F}}_p) = 0$.*

In 1941, Deuring ([3]) determined, for E a supersingular elliptic curve over $\overline{\mathbb{F}}_p$, that $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is a maximal order in the quaternion algebra $B_{p,\infty}$ over $\mathbb{Q}$. It has been pointed out to me by Prof. Ernst Kani that in [4], Deuring indeed discussed the connection with Hecke's conjecture, albeit supposing wrongly that the latter held. Using the idea of $\mathfrak{A}$-transform of Serre, one can show that there is a bijection between left ideal classes $[I_1], \ldots, [I_H]$ of $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ and isomorphism classes of supersingular elliptic curves $E_1, \ldots, E_H$ over $\overline{\mathbb{F}}_p$, given functorially by the tensor map

$$[I] \mapsto [E \otimes_{\mathrm{End}(E)} I].$$

Remark 1.2. *From a modern point of view, the most natural geometrical context where supersingular elliptic curves arise is in the special fiber at p of the elliptic curve $X_0(p)$, consisting of two projective lines intersecting at supersingular points. By flatness of the model of $X_0(p)$ over $\mathrm{Spec}(\mathbb{Z})$, the number $|S|$ of supersingular points on $X_0(p)_{\overline{\mathbb{F}}_p}$ and the genus g of the Riemann surface $X_0(p)_{\mathbb{C}}$ are related by the formula $|S| = g + 1$ and once we identify modular forms and differential forms on $X_0(p)_{\mathbb{C}}$, this explains the similitary of the formulas for the dimension of $S_2(\Gamma_0(p))$, the number of supersingular points and thus the class number.*

It is not too hard to check that the norm form of $\phi \in \mathrm{End}(E)$ coming from the quaternion algebra corresponds to the degree of ϕ as an endomorphism. This holds more generally for ideals $\mathrm{Hom}(E_i, E_j)$ (i.e., isogenies $\phi : E_i \longrightarrow E_j$), and thus the above bijection can be strenghtened to include the quadratic module structure.
We are now in position to give the geometric interpretation of Eichler's original Basis Problem.

Proposition 1.3. *The theta series coming from the modules*

$$\mathrm{Hom}(E_i, E_j) \cong I_j^{-1} I_i$$

equipped with the quadratic degree map span *the rational vector space $S_2(\Gamma_0(p))$.*

It is worth pointing out that in 1982, Ohta ([19]) gave an explicit connection between the geometry of $X_0(p)$ in characteristic p and the basis problem modulo p. Further development of the geometric perspective can be found in Gross ([9]). As for recent work on the Eichler Basis Problem from this point of view, we cite [7] that establishes the integral version of the basis problem using deep methods and ideas of Mazur and Ribet on modular curves.
The remainder of the paper deals with the generalization of the above results to *superspecial* points (to be defined shortly) on a Hilbert moduli space.
This Hilbert moduli space is an algebraic stack parametrizing principally polarized abelian varieties with RM. Note that this moduli space is the natural generalization of $X_0(1)$, not $X_0(p)$. In particular, we use very little information about the global geometry of the space (except maybe when p is ramified).

Terminology

We explain the meaning of two concepts that are identical for elliptic curves, but decisively different for higher dimensional abelian varieties. Let k be an algebraically closed field of characteristic $p > 0$.

Definition 1.4. *A abelian variety A over k of dimension g is* superspecial *if and only if $A \cong E^g$, for E some supersingular elliptic curve.*

In dimension $g \geq 2$, there is a unique superspecial abelian variety by the following theorem.

Theorem 1.5. (Deligne [21]) *Let E_1, E_2, E_3, E_4 be supersingular elliptic curves over $k = \overline{k}$. Then $E_1 \times E_2 \cong E_3 \times E_4$.*

In dimension one, we obtain the same objects if we replace the condition $A \cong E^g$ by the condition that $A \sim E^g$ i.e., that A is merely isogenous to E^g. In higher dimensions, this is false e.g., the isogeny class of E^g contains infinitely many isomorphism classes.

Definition 1.6. *A abelian variety A over k of dimension g is* supersingular *if and only if $A \sim E^g$, for E some supersingular elliptic curve. Equivalently, all the slopes of its Newton polygon are $\frac{1}{2}$.*

The same definitions apply of course to abelian varieties with additional structures. In the RM case, the superspecial condition yields finitely many isomorphism classes (in contrast with the supersingular condition). Also, it is a fact that the number of polarizations of fixed degree (e.g., principal polarizations) on an abelian variety is finite. In particular, the superspecial locus on the Hilbert moduli space i.e., the set of points whose underlying abelian variety is superspecial, is finite. On the other hand, the supersingular locus is positive dimensional for $g > 1$.

2. Superspecial Orders in $B_{p,\infty} \otimes L$

To fix notation, we recall some basic material about quaternion algebras.

2.1. Quaternion algebras.

Let L be any field.

Definition 2.1. *A quaternion algebra B over L is a central, simple algebra of rank 4 over L.*

If the char $L \neq 2$, the quaternion algebra B is given by a couple (c, d), where $c, d \in L\backslash\{0\}$, as the L-algebra of basis $1, i, j, k$, where $i, j \in B, k = ij$, and

$$i^2 = c, \quad j^2 = d, \quad ij = -ji.$$

A quaternion algebra is equipped with a canonical involutive L-endomorphism $b \mapsto \overline{b}$ called conjugation. The (reduced) norm of B is defined as $n(b) := b\overline{b}$, $b \in B$.
Any field L admits over itself the quaternion algebra $M_2(L)$. For local fields (different than $\mathbb{C}$), there is only one more:

Theorem 2.2. *Let $L \neq \mathbb{C}$ be a local field. Then there exists a unique quaternion division algebra over L, up to isomorphism.*

Theorem 2.3. *Let B be a quaternion algebra over a number field L. Let v be a place of L. We denote $B_v := B \otimes_L L_v$. A place v is ramified if B_v is a division algebra. If $B_v \cong M_2(L_v)$, we say the place v is split.*

Theorem 2.4. *Let L be a number field. The number $|\mathbf{Ram}(B)|$ of ramified places is even. For any even set S of places, there exists a unique quaternion algebra B/L up to isomorphism such that $\mathbf{Ram}(B) = S$.*

Example 2.5. *The quaternion algebra $B_{p,\infty}$ over $\mathbb{Q}$ is ramified only at p and ∞ i.e., $B_{p,\infty} \otimes \mathbb{Q}_\ell \cong M_2(\mathbb{Q}_\ell)$ for $\ell \neq p, \infty$.*

In general, we denote by $B_{\nu_1,\cdots,\nu_{2m}}$ the quaternion algebra ramified at the places $\nu_1, \cdots, \nu_{2m}$.

2.2. **Orders.** Having recalled the rational theory of quaternion algebras, we now describe a certain class of orders of $B_{p,\infty} \otimes L$ arising from superspecial abelian varieties with real multiplication by $\mathcal{O}_L$, where L is a totally real field.

Definition 2.6. *Let B be the quaternion algebra over $L_\mathfrak{p}$. Let $K = K_\mathfrak{p}$ be a quadratic extension of $L_\mathfrak{p}$ contained in B. Set*

$$R_v(K) = \mathcal{O}_K + P_B^{v-1},$$

for P_B the unique maximal ideal in $\mathcal{O}_B$ and $v = 1, 2, \ldots$.

Definition 2.7. *An order $\mathcal{O}$ is* superspecial *of level $\mathcal{P}$ dividing p, $\mathcal{P} = \prod_i \mathfrak{p}_i^{\alpha_i} \cdot \prod_j \mathfrak{q}_j^{\beta_j}$, for $\mathfrak{p} \in Ram(B_{p,\infty} \otimes L)$, $\mathfrak{q}_j \notin Ram(B_{p,\infty} \otimes L)$, if:*

- for $\alpha_i \geq 1$, there is an unramified quadratic extension $\mathcal{O}_K$ of $\mathcal{O}_{L_\mathfrak{p}}$ such that $\mathcal{O}_{\mathfrak{p}_i} = R_{\alpha_i}(K)$;
- for $\beta_j > 1$, if $f(\mathfrak{q}_j/p)$ is even, $\mathcal{O}_{q_j}$ contains a split quadratic extension; if $f(\mathfrak{q}_j/p)$ is odd, there is an unramified quadratic extension $\mathcal{O}_K$ such that $\mathcal{O}_{\mathfrak{q}_j} \cong \left\{ \begin{pmatrix} \alpha & \beta^\sigma \\ \pi_{\mathfrak{q}_j}^{\beta_j}\beta & \alpha^\sigma \end{pmatrix}, \alpha, \beta \in \mathcal{O}_K \right\}$, for σ the involution on K, $\pi_{\mathfrak{q}_j}$ a uniformizer in $\mathcal{O}_{L_{\mathfrak{q}_j}}$;
- for any other finite prime $\mathfrak{l}$, $\mathcal{O}_\mathfrak{l}$ contains a split extension (i.e., $\mathcal{O}_{L_\mathfrak{l}} \oplus \mathcal{O}_{L_\mathfrak{l}}$).

We will explain later on how superspecial orders arise as endomorphism orders $\mathrm{End}_{\mathcal{O}_L}(A)$ of superspecial abelian varieties A with RM.

Example 2.8. *Let p be unramified. Then a superspecial order of level p is an Eichler order i.e., the intersection of two maximal orders (not necessarily distinct). This follows from the facts that p is squarefree and being Eichler is a local property.*

3. The Basis Problem for Hilbert Modular Forms

The Jacquet-Langlands correspondence implies the following classical statement.

Theorem 3.1. *Let p unramified. Let $S_2(\Gamma_0(p), 1)^{new}$ be the subspace of new-forms of the vector space of Hilbert modular forms of weight two, level p. Then $S_2(\Gamma_0(p), 1)^{new}$ is spanned by differences of theta series coming from left ideals of an Eichler order of level p in the quaternion algebra $B_{p,\infty} \otimes L$.*

4. Geometric interpretation

In this section, we explain the origin of the concept of a superspecial order (cf. Definition 2.7) and we give a geometric interpretation of the quadratic modules giving rise to theta series.

Theorem 4.1. *For any superspecial abelian variety A with RM by $\mathcal{O}_L$, the endomorphism order $\mathrm{End}_{\mathcal{O}_L}(A)$ is a superspecial order.*

Proof. (Sketch for p unramified)

Let A be an abelian variety defined over $\overline{\mathbb{F}}_p$. For a rational prime $\ell \neq p$, we let $T_\ell(A) = \varprojlim A[\ell^n]$ i.e., the Tate module at ℓ. At $\ell = p$, let $\mathbb{D}(A)$ be the Dieudonné module (cf. Section 5 for details). Then we have the well-known RM version of Tate's theorem (where the finite field k is such that A_1, A_2 and all $\mathcal{O}_L$-homomorphisms are defined over it):

Theorem 4.2. *Let A_1, A_2 be two supersingular abelian varieties with RM by $\mathcal{O}_L$. Then for $\ell \neq p$,*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_L,k}(A_1, A_2) \otimes \mathbb{Z}_\ell &\cong \mathrm{Hom}_{\mathcal{O}_L \otimes \mathbb{Z}_\ell}(T_\ell(A_1), T_\ell(A_2)) \\ &\cong M_2(\mathcal{O}_L \otimes \mathbb{Z}_\ell), \\ \mathrm{Hom}_{\mathcal{O}_L,k}(A_1, A_2) \otimes \mathbb{Z}_p &\cong \mathrm{Hom}_{\mathcal{O}_L \otimes W(k)[F,V]}(\mathbb{D}(A_2), \mathbb{D}(A_1)), \end{aligned}$$

where the homomorphisms respect the $\mathcal{O}_L$-structures.

Since local deformation theory decomposes according to primes, p unramified, implies there is a unique isomorphism class of Dieudonné module $\mathbb{D}$ with RM by reduction to the inert case. We can thus pick any point that we like to compute the discriminant of the order e.g., the superspecial abelian variety $E \otimes_{\mathbb{Z}} \mathcal{O}_L$. Since

$$\mathrm{End}_{\mathcal{O}_L}(E \otimes_{\mathbb{Z}} \mathcal{O}_L) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathcal{O}_L,$$

we find that it is $p\mathcal{O}_L$, since the discriminant of the order $\mathrm{End}(E)$ is p, since it is maximal in $B_{p,\infty}$.

□

Theorem 4.3. *Let $h^+(L) = 1$. Let p be unramified. Let A be a superspecial abelian variety with RM satisfying the Rapoport condition. The map $I \mapsto A \otimes_{\mathcal{O}} I$ induces a functorial bijection from left ideal classes of $\mathcal{O}$ to superspecial abelian varieties with RM satisfying the Rapoport condition.*

Example 4.4. *Let $g = 2$. Let A be a superspecial abelian surface with RM by $\mathcal{O}_L$.*

p	$\mathrm{End}_{\mathcal{O}_L}(A)$	$B_{p,\infty} \otimes L$
inert: $p = \mathfrak{p}$	*Eichler of level* p	B_{∞_1,∞_2}
split: $p = \mathfrak{p} \cdot \bar{\mathfrak{p}}$	*maximal*	$B_{\mathfrak{p},\bar{\mathfrak{p}},\infty_1,\infty_2}$
ramified: $p = \mathfrak{p}^2$	• *maximal of level* 1 • *superspecial of level* $\mathfrak{p}^2$	B_{∞_1,∞_2}

So far, we provided a geometric interpretation of projective modules of superspecial orders as modules of $\mathcal{O}_L$-isogenies $\mathrm{Hom}_{\mathcal{O}_L}(A_i, A_j)$. We now explain how these latter modules can be also be given a quadratic module structure in a natural way by using the geometry of the abelian varieties. For (A_1, λ_1), (A_2, λ_2), two principally polarized superspecial abelian varieties and $\phi \in \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)$, define

$$||\phi||_{\mathcal{O}_L} = ||\phi||_{\mathcal{O}_L} := \lambda_1^{-1} \circ \phi^t \circ \lambda_2 \circ \phi, \qquad \begin{array}{ccc} A_2^t & \xleftarrow{\lambda_2} & A_2 \\ {\scriptstyle \phi^t}\downarrow & & \uparrow{\scriptstyle \phi} \\ A_1^t & \xrightarrow[\lambda_1^{-1}]{} & A_1 \end{array} .$$

The application $||-||_{\mathcal{O}_L}$ is an $\mathcal{O}_L$-integral quadratic form:

$$||-||_{\mathcal{O}_L} : \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2) \longrightarrow \mathrm{End}_{\mathcal{O}_L}(A_1)^{R=1} = \mathcal{O}_L.$$

The only non-trivial fact that needs to be checked is that it indeed takes values in $\mathcal{O}_L$. This holds because the formula $\lambda_1^{-1} \circ \phi^t \circ \lambda_2 \circ \phi$ is stable under the Rosati involution, which is simply the canonical involution of the totally definite quaternion algebra $B_{p,\infty} \otimes L$.

The theta series

$$\Theta(\mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)) := \sum_{\mathcal{O}_L \ni \nu \gg 0 \text{ or } \nu = 0} a_\nu q^\nu,$$

where $a_\nu = |\{\phi \in \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2) \text{ such that } ||\phi||_{\mathcal{O}_L} = \nu\}|$, is the q-expansion of a Hilbert modular form of parallel weight 2 for the group $\Gamma_0\big((p)\big) \subset \mathrm{SL}_2(\mathcal{O}_L)$.
We can now state the geometric interpretation of Eichler's Basis Problem for Hilbert modular forms:

Theorem 4.5. *Let $h^+(L) = 1$, and p be unramified. Let $(A_i, \iota_i, \lambda_i)$ run through the superspecial points on the Hilbert moduli space. The differences of theta series coming from the quadratic modules*

$$(\mathrm{Hom}_{\mathcal{O}_L}(A_i, A_j), || - ||_{\mathcal{O}_L})$$

span *the vector space $S_2^{new}(\Gamma_0((p)))$ of Hilbert modular newforms.*

A similar theorem holds for p totally ramified.

5. Classification up to isomorphism of Dieudonné modules over totally ramified Witt vectors

Let A be a superspecial abelian variety with RM by $\mathcal{O}_L$ over a perfect field k. When p is ramified in $\mathcal{O}_L$, it is not true that the order $\mathrm{End}_{\mathcal{O}_L}(A)$ always has level p. This is related to the number of isomorphism classes of superspecial Dieudonné modules with RM being in general greater than one (in contrast with the unramified case, cf. the proof of Theorem 4.1). Dieudonné modules arise in geometry in a way relevant to us as the first crystalline cohomology group $H^1_{cris}(A/W(k))$ of an abelian variety $A_{/k}$. Since this construction is functorial, additional structure (such as real multiplication) carry over from A to the Dieudonné module. In this section, we thus sketch the classification up to isomorphism of Dieudonné modules over *totally ramified* Witt vectors (our proof in [18] follows Manin ([16]) mutatis mutandis).

Let k be algebraically closed, and let $\mathfrak{F}$ be a totally ramified extension of $\mathbb{Q}_p$. The Witt vectors $W(k)$ is a complete discrete valuation ring in characteristic zero with residue field k i.e., $W(k)/pW(k) \cong k$. Let K be the fraction field of $W(k)$. Denote by $K_{\mathfrak{F}} := K \cdot \mathfrak{F}$ the compositum of K and $\mathfrak{F}$, with ring of integers $W_{\mathfrak{F}}$. The main tools that appear in Manin's classification are two finiteness theorems and some algebro-geometric classifying spaces. The key idea behind the finiteness theorems is the concept of a *special* module (due to Remark 5.6, we refer the reader to [**?**, Def. 1.3.11] for the definition); a crucial fact is that every Dieudonné module has a unique maximal special submodule, of finite colength.

Definition 5.1. *A Dieudonné module $\mathbb{D}$ is a left $W_{\mathfrak{F}}[F, V]$-module free of finite rank over $W_{\mathfrak{F}}$ with the condition that $\mathbb{D}/F\mathbb{D}$ has finite length.*

Definition 5.2. *Two Dieudonné modules $\mathbb{D}_1, \mathbb{D}_2$ are isogenous if there is an injective homomorphism $\phi : \mathbb{D}_1 \hookrightarrow \mathbb{D}_2$ such that $\mathbb{D}_2/\phi(\mathbb{D}_1)$ has finite length over $W_{\mathfrak{F}}$. If $\mathbb{D}_1$ is isogenous to $\mathbb{D}_2$, we write: $\mathbb{D}_1 \sim \mathbb{D}_2$.*

Theorem 5.3. *(First Finiteness Theorem) Let $\mathbb{D}$ be a Dieudonné module. There exists only a finite number of non-isomorphic special modules isogenous to $\mathbb{D}$.*

Theorem 5.4. *(Second Finiteness Theorem) Let $\mathbb{D}$ be a Dieudonné module. The module $\mathbb{D}$ has a maximal special submodule $\mathbb{D}_0$. The length $[\mathbb{D} : \mathbb{D}_0]$ is bounded uniformly in the isogeny class of $\mathbb{D}$.*

Theorem 5.5. *(Classification Theorem) Let k be an algebraically closed field. A Dieudonné module $\mathbb{D}$ is determined by the following collection of invariants:*

- *the Newton polygon slopes of $\mathbb{D}$;*
- *the maximal special submodule $\mathbb{D}_0 \subset \mathbb{D}$ (parametrized by discrete invariants);*
- *a $\Gamma(\mathbb{D}_0, H)$-orbit of a point corresponding to $\mathbb{D}$ in a constructible algebraic set $A(\mathbb{D}_0, H)$, where H is a nonnegative integer that depends only on the slopes; $A(\mathbb{D}_0, H)$ and $\Gamma(\mathbb{D}_0, H)$ depend only on $\mathbb{D}_0$ and H, and $\Gamma(\mathbb{D}_0, H)$ is a* finite *group.*

Two Dieudonné modules are isomorphic if and only if all these invariants coincide.

Recall that a supersingular Dieudonné module is a Dieudonné module whose Newton polygon slopes are $\frac{1}{2}$.

Remark 5.6. *A supersingular Dieudonné module is superspecial if and only if it is special.*

Corollary 5.7. *The number of isomorphism classes of superspecial Dieudonné modules with RM by $\mathcal{O}_L$ of rank 2 over a totally ramified prime $p = \mathfrak{p}^g$ is: $\left[\frac{g}{2}\right] + 1$.*

Corollary 5.8. *The levels of superspecial endomorphism orders for $p\mathcal{O}_L = \mathfrak{p}^g$ are $\{\mathfrak{p}^g, \mathfrak{p}^{g-2}, ..., \mathfrak{p}^{g-2\left[\frac{g}{2}\right]}\}$.*

5.1. **Application to Hilbert moduli spaces over totally ramified primes.** In this section, we explain that the stratification of the Hilbert moduli space over a totally ramified prime $p = \mathfrak{p}^g$ introduced by Andreatta-Goren in [1] coincides with the stratification suggested by the decomposition of the moduli spaces à la Manin, at least on the supersingular stratum.

We recall briefly the definition of the stratification of [1]. Let p be a totally ramified prime. Let A/k be a polarized abelian variety with RM, defined over a field k of characteristic p. Fix an isomorphism $\mathcal{O}_L \otimes_{\mathbb{Z}} k \cong k[T]/(T^g)$. One knows that $H^1_{dR}(A)$ is a free $k[T]/(T^g)$-module of rank 2, and there are two generators α and β such that:

$$H^1(A, O_A) = (T^i)\alpha + (T^j)\beta, i \geq j, i + j = g.$$

The index $j = j(A)$ is called the singularity index. For perspective, recall the short exact sequence:

$$0 \longrightarrow H^0(A, \Omega^1_A) \longrightarrow H^1_{dR}(A) \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow 0.$$

These modules are Dieudonné modules of group schemes, and we rewrite this exact sequence as:

$$0 \longrightarrow (k, \mathrm{Fr}^{-1}) \otimes_k \mathbb{D}(\mathrm{Ker}(\mathrm{Fr})) \longrightarrow \mathbb{D}(A[p]) \longrightarrow \mathbb{D}(\mathrm{Ker}(\mathrm{Ver})) \longrightarrow 0.$$

The slope $n = n(A)$ is defined by the relation $j(A) + n(A) = a(A)$, where $a(A)$ is the a-number of the abelian variety. The subsets $\mathfrak{W}_{(j,n)}$ parameterizing abelian varieties with singularity index j and slope n are quasi-affine, locally closed and form a stratification ([1, Thm. 10.1], [2, §6.1]). Note that for any Dieudonné module $\mathbb{D}$ with RM of rank 2, we can define abstractly $j(\mathbb{D})$ and $n(\mathbb{D})$ without any reference to abelian varieties e.g., $j(\mathbb{D}) = j$ is the integer such that

$$T^i\alpha + T^j\beta = \mathrm{Ker}(V : \mathbb{D}/p\mathbb{D} \longrightarrow \mathbb{D}/p\mathbb{D}), i \geq j,$$

for α, β some generators of $\mathbb{D}$. The slope is $n(\mathbb{D}) := a(\mathbb{D}) - j(\mathbb{D})$.
Apply the Manin classification to rank 2 modules, and label the (affine) spaces $\Gamma(\mathbb{D}_0, H)\backslash A(\mathbb{D}_0, H)$ thus obtained by $\mathfrak{M}_i$, $i \in I$, $|I| < \infty$. Define $\mathfrak{N}_i$ as the strata on the Hilbert modular variety (say, equipped with rigid level structure) such that for $\underline{A} \in \mathfrak{N}_i$, the Dieudonné module $\mathbb{D}(\underline{A})$ belongs to $\mathfrak{M}_i$. The strata $\mathfrak{N}_i$ form the so-called Manin stratification.

Theorem 5.9. *Let $p\mathcal{O}_L = \mathfrak{p}^g$. The Manin stratification of Hilbert modular varieties coincide with the slope stratification $\{\mathfrak{W}_{(j,n)}\}$.*

Remark 5.10. *It follows from the observation that $\mathfrak{M}_i$ are affine that the strata $\mathfrak{N}_i$ are, in turn,* relatively *affine i.e., $\mathfrak{N}_i$ is affine over the Hilbert modular variety.*

5.2. **The Doi-Naganuma lift.** The Doi-Naganuma lift is historically one of the first examples of the functoriality principle of Langlands, who himself generalized it to $GL(2)$ in the context of representation theory. We do not stray from the classical language.s
Let $L = \mathbb{Q}(\sqrt{p})$, with $p = 1 \mod 4$ and $h_L = 1$. We are interested in the space of Hilbert modular forms of level 1 i.e., for the full modular group $\mathrm{SL}_2(\mathcal{O}_L)$.

Let k be an even weight. Let $S_k(\Gamma_0(p), \chi_p)$ be the space of holomorphic modular cuspforms of weight k non-trivial quadratic character χ_p, also called modular forms of nebentypus. A form $f \in S_k(\Gamma_0(p), \chi_p)$ has a Fourier expansion:

$$f(\tau) = \sum_{n\geq 0} c(n)q^n, \quad q = e^{2\pi i\tau}.$$

Let us define the subspaces $+$ and $-$ of $S_k(\Gamma_0(p), \chi_p)$ by:

$$S_k^{\pm}(\Gamma_0, \chi_p) = \{f \in S_k(\Gamma_0(p), \chi_p) | \chi_p(n) = \mp 1 \implies c(n) = 0\}.$$

It is a lemma of Hecke that the space $S_k(\Gamma_0(p), \chi_p)$ decomposes as a direct sum:

$$S_k(\Gamma_0(p), \chi_p) = S_k^+(\Gamma_0(p), \chi_p) \oplus S_k^-(\Gamma_0(p), \chi_p).$$

Theorem 5.11. *Let $f(\tau) = \sum_{n\geq 0} c(n)q^n \in S_k^+(\Gamma_0(p), \chi_p)$. There exist a Hilbert cuspform $\Phi(f)$ of weight k for $\mathrm{SL}_2(\mathcal{O}_L)$ of Fourier expansion:*

$$\Phi(f)(z_1, z_2) = \frac{-B_k}{2k}\tilde{c}(0) + \sum_{\nu \in \mathcal{D}_L^{-1}, v \gg 0} \sum_{d|\nu} d^{k-1}\tilde{c}\Big(\frac{p\nu\overline{\nu}}{d^2}\Big) q_1^{\nu} q_2^{\overline{\nu}},$$

where B_k is the k-th Bernouilli number, $q_j = e^{2\pi i z_j}$, and $\tilde{c}(n) = \begin{cases} c(n), & p \nmid n \\ 2c(n), & p | n \end{cases}$.

The Doi-Naganuma map is the application:

$$DN : S_k(\Gamma_0(p), \chi_p) \longrightarrow S_{k,k}(\mathrm{SL}_2(\mathcal{O}_L)),$$

defined as follows: $DN(f) := \begin{cases} \Phi(f), & f \in S_k^+(\Gamma_0(p), \chi_p) \\ 0, & f \in S_k^-(\Gamma_0(p), \chi_p) \end{cases}$
This application is compatible with the Hecke algebra actions: Hecke eigenforms are sent to Hecke eigenforms.

Proposition 5.12. *(Doi-Naganuma) There is an isomorphism of Hecke modules:*

$$S_k^+(\Gamma_0(p), \chi_p) \oplus S_k(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\cong} S_{k,k}(\mathrm{SL}_2(\mathcal{O}_L))^{sym},$$

where ^{sym} *denotes the symetric Hilbert modular forms i.e., $F(x, y) = F(y, x)$.*

Note that for $k < 12$, $S_k(\mathrm{SL}_2(\mathbb{Z})) = 0$. In particular,

$$S_2^+(\Gamma_0(p), \chi_p) \overset{DN}{\cong} S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L))^{sym}.$$

Example: Let $k = 2$. The Doi-Naganuma admits a geometric interpretation in characteristic $p > 0$ at primes of bad reduction.
The key observation is to retrieve the space $S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L))$ directly on the singular Hilbert modular surface. The singular locus of the Hilbert modular surfaces at the ramified prime p is made of superspecial points A_i whose endomorphisms orders are maximal in B_{∞_1,∞_2}. We can compute the ℓ-adic character group of the Hilbert modular surface using vanishing cycles and we thus retrieve the space $S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L))$ after tensoring with $\mathbb{C}$. More simply put, differences of theta series constructed from $\mathcal{O}_L$-modules of isogenies $\mathrm{Hom}_{\mathcal{O}_L}(A_i, A_j)$ span $S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L))$.
N.B. This can be pushed slightly further: under $h(L) = 1$, we can retrieve algebraically (and thus geometrically by the above) the symmetric Hilbert modular forms (cf. works of Ponomarev, Waldspurger, Vignéras, etc.). This might be viewed as a bad reduction version of the pioneering work of Hirzebruch and Zagier in characteristic zero. At the time of writing these notes, I do not know if an elegant chararacteristic p version encompassing the modular curve $X_1(p)$ (to retrieve the space $S_2(\Gamma_0(p), \chi_p)$) can also be developped.

References

[1] Andreatta, F., Goren, E.Z., Geometry of Hilbert Modular Varieties over Totally Ramified Primes, *Int. Math. Res. Not.,* 2003, no. **33**, 1786–1835.

[2] Andreatta, F., Goren, E.Z., Hilbert modular varieties of low dimension. *Geometric aspects of Dwork theory,* Vol. **I**, 113–175, Walter de Gruyter, Berlin, 2004.

[3] Deuring, M., Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, *Abh. Math. Sem. Hansischen Univ.* **14**, (1941). 197–272.

[4] Deuring, M., Die Anzahl der Typen von Maximalordnungen einer definiten Quaternionenalgebra mit primer Grundzahl, *Jber. Deutsch. Math. Verein.*, **54**, (1950). 24–41.

[5] Eichler, M., Über die Darstellbarkeit von Modulformen durch Thetareihen, *J. Reine Angew. Math.*, **195**, (1955), 156–171.

[6] Eichler, M., The basis problem for modular forms and the traces of the Hecke operators, *Modular functions of one variable, I* , (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 75–151, Lecture Notes in Math., Vol. **320**, Springer, Berlin, 1973.

[7] Emerton, M., Supersingular elliptic curves, theta series and weight two modular forms, *J. Amer. Math. Soc.,* **15,** (2002), no. 3, 671–714.

[8] Gelbart, S., *Automorphic forms on adele groups,* Annals of Mathematics Studies, No. **83**, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo, 1975. x+267 pp.

[9] Gross, B., Heights and the special values of L-series, Number theory (Montréal, Qué., 1985), 115–187, *CMS Conf. Proc.*, **7**, Amer. Math. Soc., Providence, RI, 1987.

[10] Hecke, E., *Mathematische Werke,* Herausgegeben im Auftrage der Akademie der Wissenschaften zu Göttingen Vandenhoeck und Ruprecht, Göttingen, 1959, 955 pp.

[11] Hida, H., The integral basis problem of Eichler, *Int. Math. Res. Not.*, 2005, no. **34**, 2101–2122.

[12] Hijikata, H.; Pizer, A.; Shemanske, T., Orders in quaternion algebras, *J. Reine Angew. Math.,* **394,** (1989), 59–106.

[13] Hijikata, H.; Pizer, A.; Shemanske, T., *The basis problem for modular forms on* $\Gamma_0(N)$*,* Mem. Amer. Math. Soc. **82,** (1989), no. 418, vi+159 pp.

[14] Hirzebruch, F.; Zagier, D., Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, *Invent. Math.,* **36,** (1976), 57–113.

[15] Jacquet, H.; Langlands, R. P., *Automorphic forms on* GL(2), Lecture Notes in Mathematics, Vol. **114**, Springer-Verlag, Berlin-New York, 1970. vii+548 pp.

[16] Manin, Y., Theory of commutative formal groups over fields of finite characteristic, *Uspehi Mat. Nauk,* **18**, 1963, no. 6, (114), 3–90.
[17] Nicole, M.-H., Superspecial Abelian Varieties and the Eichler Basis Problem for Hilbert Modular Forms, J.N.T., 2008.
[18] Nicole, M.-H., The Manin Stratification of Hilbert Modular Varieties, preprint, 2008.
[19] Ohta, M., On theta series mod p, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.,* **28**, (1981), no. 3, 679–686 (1982).
[20] Pizer, A., An algorithm for computing modular forms on $\Gamma_0(N)$, *J. Algebra,* **64**, (1980), no. 2, 340–390.
[21] Shioda, T., Supersingular $K3$ surfaces, *Algebraic geometry,* (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 564–591, Lecture Notes in Math., **732**, Springer, Berlin, 1979.
[22] Tate, J., Endomorphisms of abelian varieties over finite fields, *Invent. Math.* **2,** 1966, 134–144. Waterhouse, W., Abelian varieties over finite fields, *Ann. Sci. Ecole Norm. Sup.* (4), **2**, 1969, 521–560.

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