SHIMURA CURVES II

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ABSTRACT. These are the notes of a talk I gave at the number theory seminar at University of Duisburg-Essen in summer 2008.

We discuss the adèlic description of quaternionic Shimura curves. The adèlic description has four advantages over the classical description of Shimura curves discussed in the previous talk. Namely, the adèlic language is the right one in view towards the Langlands program, we don't need an integral structure, it makes certain adèlic actions on the Shimura curve explicit and we can work uniformly over smaller fields of definition.

CONTENTS

1. THE ALGEBRAIC GROUP G ASSOCIATED TO A QUATERNION ALGEBRA

Setup 1.1. We are given the following data:

- F a totally real number field of degree $[F : \mathbb{Q}] = n$
- B/F a quaternion algebra, i. e. B/F is a central, simple algebra of $\dim_F B = 4$
- B/F splits at exactly one infinite place (not used until Section 2)

Example 1.2. $B = M_2(F)$.

Definition 1.3 ($B^\times$ considered as algebraic group)**.** The algebraic group G'/F representing the functor

$$\begin{array}{ccc} (F\text{-algebras}) & \longrightarrow & (\text{groups}) \\ R & \longmapsto & (B \otimes_F R)^\times \end{array}$$

is called the *algebraic group over F given by $B^\times$*.

Example 1.4. For $F = \mathbb{Q}$ and $B = M_2(\mathbb{Q})$, we have $B^\times = \mathrm{GL}_2(\mathbb{Q})$ so that $G' = \mathrm{GL}_{2,\mathbb{Q}}$ is the general linear group.

Remark 1.5. We show that G'/F exists. Let B/F be a quaternion algebra.
$\Rightarrow$ $B = \left(\frac{a,b}{F}\right)$, i. e. $\exists\ a, b \in F$ such that as an F-algebra $B = F[\alpha, \beta]$ is generated by two elements $\alpha, \beta \in B$ subject to the relations

$$\alpha^2 = a,\ \beta^2 = b,\ \alpha\beta = -\beta\alpha.$$

Let $\gamma \in B$ be an arbitrary element.
$\Rightarrow$ we can write γ in a unique way as

$$\gamma = x \cdot 1 + y \cdot \alpha + z \cdot \beta + w \cdot \alpha\beta$$

with $x, y, z, w \in F$.
This suggests that G' is a subscheme of $\mathbb{A}^4_F$ by

$$G' \subset \mathbb{A}^4_F, \quad \gamma \mapsto \underline{\gamma} := (x, y, z, w).$$

To really get the units $B^\times$ of B, we need an algebraic description of them. Consider the reduced norm of B

$$\mathrm{nrd}(\gamma) = \gamma \cdot \bar{\gamma} = \underbrace{x^2 - ay^2 - bz^2 + abw^2}_{=: f(x,y,z,w) \in F[x,y,z,w]}$$

where $\bar{\gamma} = x \cdot 1 - y \cdot \alpha - z \cdot \beta - w \cdot \alpha\beta$ is the conjugation on B. We easily see that

$$\gamma \in B^\times \iff f(x, y, z, w) = \mathrm{nrd}(\gamma) \neq 0.$$

Hence, we set

$$\begin{aligned} G' &:= \operatorname{Spec} F[x, y, z, w]_{f(x,y,z,w)} \\ &\simeq \operatorname{Spec} F[x, y, z, w, u]/\big(u \cdot f(x, y, z, w) - 1\big). \end{aligned}$$

The algebraic map defined over F

$$G' \times G' \to G',\ \big(\underline{\gamma}, \underline{\gamma'}\big) \mapsto \underline{\gamma \cdot \gamma'}$$

makes G' an algebraic group over F which fulfills the required property

$$G'(R) = (B \otimes_F R)^\times$$

for any F-algebra R.

Example 1.6. If $B = M_2(F)$, then $\mathrm{nrd}(\gamma) = \det(\gamma)$ so that $G' = \mathrm{GL}_{2,F}$.

We want that the group G'/F lives over $\mathbb{Q}$. Therefore, we take the Weil restriction of G'. References for the Weil restriction are [BLR90] and [Po03].

Definition 1.7 (The Weil restriction of G')**.** For the algebraic group G'/F given by B/F, let

$$G := \mathrm{Res}_{F/\mathbb{Q}}\, G'$$

be the *Weil restriction* of G' from F to $\mathbb{Q}$, i. e. G is the algebraic group over $\mathbb{Q}$ that represents the functor

$$\begin{array}{ccc} (\mathbb{Q}\text{-algebras}) & \longrightarrow & (\text{groups}) \\ R & \longmapsto & G'(R \otimes_{\mathbb{Q}} F) = (B \otimes_{\mathbb{Q}} R)^\times \end{array}$$

Remark 1.8. Since G' is affine, G will also be affine. We have

$$G(\mathbb{Q}) = B^\times \quad \text{and} \quad G(F) = \prod_{i=1}^{[F:\mathbb{Q}]} B^\times.$$

In the literature, the group G is often denoted by "$\mathrm{Res}_{F/\mathbb{Q}}\, B^\times$".

Example 1.9 (For the Weil restriction – the real torus)**.** We discuss an example for the Weil restriction of an affine algebraic group. Since a quaternionic example would be too long, we discuss the easier example of

$$\mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}\ \mathbb{C}[U,V]/(UV-1)$$

restricted to $\mathbb{R}$. Hence, let

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\ \mathbb{G}_{m,\mathbb{C}}$$

be the *real torus*. This example is not only easier, it will also be useful later on. Now, take the $\mathbb{R}$-basis $\{1, i\}$ of $\mathbb{C}$ and introduce new variables U_1, U_2, V_1, V_2. Set

$$U = U_1 \cdot 1 + U_2 \cdot i \quad \text{and} \quad V = V_1 \cdot 1 + V_2 \cdot i.$$

and express the defining equation for $\mathbb{G}_{m,\mathbb{C}}$ in the new variables with respect to the basis $\{1, i\}$, i. e.

$$\begin{aligned} UV - 1 &= (U_1 + U_2 \cdot i)(V_1 + V_2 \cdot i) - 1 \\ &= \underbrace{(U_1V_1 - U_2V_2 - 1)}_{=:f_1} \cdot 1 + \underbrace{(U_1V_2 + U_2V_1)}_{=:f_2} \cdot i. \end{aligned}$$

Then f_1, f_2 are the defining equations for $\mathbb{S}$, i. e.

$$\begin{aligned} \mathbb{S} &= \mathrm{Spec}\ \mathbb{R}[U_1, U_2, V_1, V_2]/(U_1V_1 - U_2V_2 - 1, U_1V_2 + U_2V_1) \\ &\simeq \mathrm{Spec}\ \mathbb{R}[X,Y,T]/\big(T(X^2+Y^2) - 1\big) \end{aligned}$$

where the last isomorphism is given by

$$X \mapsto U_1,\ Y \mapsto U_2,\ T \mapsto V_1^2 + V_2^2.$$

From the second represenatation of $\mathbb{S}$, one easily sees that $\mathbb{S}$ fulfills the properties of the Weil restriction. In particular, we have $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$.

It is a little bit more difficult to determine the group law of the Weil restriction. This we will not discuss. In our example of the real torus, the group law with respect to the second representation is given by

$$\begin{array}{ccc} \mathbb{S} \times \mathbb{S} & \longrightarrow & \mathbb{S} \\ \big((x,y,t),(x',y',t')\big) & \longmapsto & (xx' - yy', xy' + yx', tt') \end{array}$$

which is clearly an algebraic map.

Remark 1.10. If $F = \mathbb{Q}$, then $G = G'$.

2. The space X

Assumption 2.1. B/F splits at exactly one infinite place of F. This assumption ensures that we will get a Shimura **curve**. If B splits at more than one infinite place, then we will get a higher-dimensional Shimura variety.

Remark 2.2. Let $\{\tau_1, \ldots, \tau_n\}$ be the set of real embeddings $\tau_i : F \hookrightarrow \mathbb{R}$ of F. Then by the above assumption, the real points of the group G are

$$\begin{aligned} G(\mathbb{R}) &= (B \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \\ &= \big(B \otimes_F (F \otimes_{\mathbb{Q}} \mathbb{R})\big)^{\times} \\ &= \prod_{\tau_i : F \hookrightarrow \mathbb{R}} (B \otimes_{\tau_i} \mathbb{R})^{\times} \\ &= \mathrm{GL}_2(\mathbb{R}) \times \underbrace{\mathbb{H}^{\times} \times \cdots \times \mathbb{H}^{\times}}_{[F:\mathbb{Q}]-1 \text{ times}} \end{aligned}$$

where $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ denotes the Hamiltionian quaternions over $\mathbb{R}$.

Remark 2.3. $G(\mathbb{R})$ acts on the set $\mathrm{Hom}_R(\mathbb{S}, G_{\mathbb{R}})$ by conjugation, i. e. the map

$$\begin{array}{ccc} G(\mathbb{R}) \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}}) & \longrightarrow & \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}}) \\ (\ r\ ,\ h(-)\ ) & \longmapsto & r \cdot h(-) \cdot r^{-1} \end{array}$$

defines a group action.

Definition 2.4 (The space X)**.** We define the space

$$X := G(\mathbb{R})\text{-orbit of } h : \mathbb{S} \to G_{\mathbb{R}}$$

where the map $h \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})$ is given by

$$h : \mathbb{S} \longrightarrow G_{\mathbb{R}}, \quad x + y \cdot i \longmapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, 1, \ldots, 1 \right)$$

Remark 2.5. The action of $G(\mathbb{R})$ on X factors through the quotient $\mathrm{GL}_2(\mathbb{R})$. $\mathrm{GL}_2(\mathbb{R})$ also acts on the space $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ by fractional linear transformations. In fact, these two spaces are the same.

Proposition 2.6. *There is an isomorphism of* $\mathrm{GL}_2(\mathbb{R})$*-sets*

$$X \simeq \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$$

sending the map h to i. In particular, X has a decomposition $X = X^+ \amalg X^-$ where X^+ denotes the component containing h so that X^+ is identified with the upper half-plane while X^- corresponds to the lower half-plane.

Proof. We have the following identifications of $\mathrm{GL}_2(\mathbb{R})$-sets.

Step 1: X and representations of $\mathbb{S}$.
There is a 1-1-correspondence

$$\begin{array}{ccc} X & \overset{1:1}{\longleftrightarrow} & \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, \mathrm{GL}_{2,\mathbb{R}}) \\ h & \longmapsto & pr \circ h \\ \big[s \mapsto \big(h(s), 1, \dots, 1\big)\big] & \longleftarrow & h \end{array}$$

where $pr : G_{\mathbb{R}} \to \mathrm{GL}_{2,\mathbb{R}}$ is the projection. The $G(\mathbb{R})$-action by conjugation on X translates into the $\mathrm{GL}_2(\mathbb{R})$-action by conjugation on the set $\mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, \mathrm{GL}_{2,\mathbb{R}})$.

Step 2: Representations of $\mathbb{S}$ and complex structures.
There is a 1-1-correspondence

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, \mathrm{GL}_{2,\mathbb{R}}) & \overset{1:1}{\longleftrightarrow} & \left\{\begin{array}{l}\text{complex structures on } V = \mathbb{R}^2, \text{ i.e. maps} \\ J \in \mathrm{End}_{\mathbb{R}}(V) \text{ which satisfy } J^2 = -1\end{array}\right\} \\ h & \longmapsto & J = h(i) \in \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{End}_{\mathbb{R}}(V) \\ \big[x + yi \mapsto x + yJ\big] & \longleftarrow & J \end{array}$$

The $\mathrm{GL}_2(\mathbb{R})$-action by conjugation on the left side becomes the $\mathrm{GL}_2(\mathbb{R})$-action by conjugation on $\mathrm{End}_{\mathbb{R}}(V)$ on the right side.

Step 3: Complex structures and Hodge structures.
There is a 1-1-correspondence

$$\begin{array}{ccc} \left\{\begin{array}{l}\text{complex structures on } V = \mathbb{R}^2, \text{ i.e. maps} \\ J \in \mathrm{End}_{\mathbb{R}}(V) \text{ which satisfy } J^2 = -1\end{array}\right\} & \overset{1:1}{\longleftrightarrow} & \left\{\begin{array}{l}\text{real Hodge structures on } V, \text{ i.e. decompositions} \\ V_{\mathbb{C}} = V^+ \oplus V^- \text{ with } \overline{V^+} = V^-\end{array}\right\} \\ J & \longmapsto & \big(\mathrm{Eig}(J_{\mathbb{C}}, +i), \mathrm{Eig}(J_{\mathbb{C}}, -i)\big) \\ \big((+i) \cdot \mathrm{id}_{V^+} \oplus (-i) \cdot \mathrm{id}_{V^-}\big)\big|_V & \longleftarrow & (V^+, V^-) \end{array}$$

where the endomorphism $(+i) \cdot \mathrm{id}_{V^+} \oplus (-i) \cdot \mathrm{id}_{V^-}$ on $V_{\mathbb{C}}$ is restricted to V via the canonical map $V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}$. The $\mathrm{GL}_2(\mathbb{R})$-action by conjugation on the complex structures becomes the $\mathrm{GL}_2(\mathbb{R})$-action as endomorphisms of $V_{\mathbb{C}}$ on the real Hodge structures, i. e.

$$\gamma \cdot (V^+, V^-) = \big(\gamma(V^+), \gamma(V^-)\big)$$

for any $\gamma \in \mathrm{GL}_2(\mathbb{R})$, where $\gamma(V^{\pm})$ denotes the image of $V^{\pm}$ under γ considered as an element of $\mathrm{GL}_2(\mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(V_{\mathbb{C}})$. Since γ is a real map, it will commute with complex conjugation so that it really maps a real Hodge structure to a real Hodge structure.

Step 4: Hodge structures and nonreal lines.
There is a 1-1-correspondence

$$\left\{\begin{array}{l}\text{real Hodge structures}\\ \text{on } V\text{, i.e. decomposi-}\\ \text{tions } V_{\mathbb{C}} = V^+ \oplus V^-\\ \text{with } \overline{V^+} = V^-\end{array}\right\} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{\begin{array}{l}\text{nonreal lines in } V_{\mathbb{C}},\\ \text{i.e. 1-dimensional sub-}\\ \text{spaces } L \subset V_{\mathbb{C}} \text{ satisfy-}\\ \text{ing } \overline{L} \neq L\end{array}\right\}$$

$$\begin{array}{ccc} (V^+, V^-) & \longmapsto & V^+ \\ (L, \overline{L}) & \longleftarrow\!\!\shortmid & L \end{array}$$

The $\mathrm{GL}_2(\mathbb{R})$-action as endomorphisms on the real Hodge structures becomes the $\mathrm{GL}_2(\mathbb{R})$-action as endomorphisms on the nonreal lines, i.e.

$$\gamma \cdot L = \gamma(L)$$

where $\gamma(L)$ is the image of L under $\gamma \in \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{C})$.

Step 5: Lines and projective space.
There is a 1-1-correspondence

$$\begin{array}{ccc} \{\text{nonreal lines in } V_{\mathbb{C}}\} & \overset{1:1}{\longleftrightarrow} & \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \\ L = \mathbb{C} \cdot \binom{x}{y} & \longleftrightarrow & [x : y] \end{array}$$

In fact, $\mathbb{P}^1(\mathbb{C})$ is by definition the set of lines in $V_{\mathbb{C}}$ while $\mathbb{P}^1(\mathbb{R})$ is given by the real lines.

The action of $\mathrm{GL}_2(\mathbb{R})$ as endomorphisms on the lines in $V_{\mathbb{C}}$ corresponds to the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{C})$ by linear fractional transformations. More precisely, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ and $L = \mathbb{C} \cdot \binom{x}{y}$ a line. Then

$$\gamma(L) = \mathbb{C} \cdot \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

and this corresponds to $[ax + by : cx + dy]$. If we write $[x : y] = [z : 1]$ (note that $[x : 0] = [1 : 0]$ is a real point), then we see that γ acts on $z \in \mathbb{C}$ as the fractional linear transformation $\frac{az+b}{cz+d}$, as claimed.

Step 6: Following the path of h.
If we follow the way of our given map

$$h : \mathbb{S} \longrightarrow G_{\mathbb{R}}, \quad x + y \cdot i \longmapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, 1, \ldots, 1 \right)$$

through the above identifications, then we see that, in the 1st step, h is sent to the map

$$pr \circ h : \mathbb{S} \to \mathrm{GL}_2(\mathbb{R}),\ x + yi \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

In the 2nd step, we get the complex structure

$$J = pr \circ h(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{End}_{\mathbb{R}}(V)$$

whose eigenspaces are

$$\mathrm{Eig}(J,+i) = \mathbb{C}\cdot\begin{pmatrix}1\\-i\end{pmatrix} \quad \text{and} \quad \mathrm{Eig}(J,-i) = \mathbb{C}\cdot\begin{pmatrix}1\\i\end{pmatrix}$$

giving the real Hodge structure

$$(V^+,V^-) = \left(\mathbb{C}\cdot\begin{pmatrix}1\\-i\end{pmatrix}, \mathbb{C}\cdot\begin{pmatrix}1\\i\end{pmatrix}\right)$$

in the 3rd step. The 4th step sends this structure to the line

$$L = V^+ = \mathbb{C}\cdot\begin{pmatrix}1\\-i\end{pmatrix}$$

which corresponds in the 5th step to the point

$$[1:-i] \quad \text{or} \quad \frac{1}{-i} = i$$

of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.

This proves the proposition. □

3. The Shimura curve $M(G,X)$

Let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$. Consider the product space $X \times G(\mathbb{A}_f)$ and let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then there is a right action of K on $X \times G(\mathbb{A}_f)$ by right multiplication on $G(\mathbb{A}_f)$ and by acting trivially on X. Further, we have a left action of $G(\mathbb{Q})$ on $X \times G(\mathbb{A}_f)$ by $G(\mathbb{Q})$ acting on $G(\mathbb{A}_f)$ by left multiplication and acting on X by conjugation. More precisely, the two actions are given by the map

$$\begin{array}{ccc} G(\mathbb{Q}) \times \big(X \times G(\mathbb{A}_f)\big) \times K & \longrightarrow & X \times G(\mathbb{A}_f) \\ \big(q, \quad (x,a), \quad k\big) & \longmapsto & q\cdot(x,a)\cdot k := (qx, qak) \end{array}$$

where $qx = q \cdot x \cdot q^{-1}$ is the action of q on x by conjugation as described in Section 2. The space of double cosets of these two actions will be our Shimura curve.

Definition 3.1 (The Shimura curve associated to K)**.** Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then

$$M_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

is called the *Shimura curve* given by (G,X,K). Since (G,X) is determined by our quaternion algebra B/F, we also call $M_K(G,X)$ the *quaternionic Shimura curve* with respect to B and K.

Example 3.2. If $B = M_2(\mathbb{Q})$, then $M_K(G,X) = \mathrm{GL}_2(\mathbb{Q})\backslash X \times \mathrm{GL}_2(\mathbb{A}_f)/K$.

Remark 3.3. The real topological group

$$G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times$$

has two connected components, since $\mathrm{GL}_2(\mathbb{R})$ consists of two components (positive and negative determinant) while $\mathbb{H}^\times$ is connected (the reduced norm on $\mathbb{H}$ is always positive, in fact $\mathrm{nrd}(\gamma) = 0 \Leftrightarrow \gamma = 0$).

Let $G(\mathbb{R})_+$ be the connected component of 1. So, an element of $G(\mathbb{R})$ is in $G(\mathbb{R})_+$ if and only if its determinant in the first component is positive. We set $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

The following theorem shows the connection of the adèlic description of a Shimura curve with the classical description as a quotient $\Gamma\backslash\mathcal{H}$ of the upper half-plane $\mathcal{H}$ by some arithmetic group Γ. Recall that in Section 2, we identified the upper half-plane $\mathcal{H}$ with X^+, the component of X containing the function h.

Theorem 3.4 (Adèlic and classical description of Shimura curves)**.** *Let $\mathcal{C}$ be a (finite) set of representatives for the double coset space $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$. Then there is a canonical isomorphism of sets*

$$M_K(G,X) \simeq \coprod_{g\in\mathcal{C}} \Gamma_g\backslash X^+$$

where $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$.

Proof. The proof is divided into several steps.

Step 1: $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ is finite.
Let $G^{ad} := G/Z$ where Z is the center of G. For example in our case, $Z \subset G$ is a torus, since quaternion algebras are central over their base field. Note that since G is reductive, this definition of the adjoint group G^{ad} coincides with the general one. Now, $G^{ad}(\mathbb{R})$ has only finitely many (namely two) connected components (see the remark above).

- $\Rightarrow$ $G^{ad}(\mathbb{R})^+\backslash G^{ad}(\mathbb{R})$ is finite, where $G^{ad}(\mathbb{R})^+$ is the connected component of one.
- $\Rightarrow$ $G(\mathbb{Q})_+\backslash G(\mathbb{Q})$ is finite, since the map $G(\mathbb{Q})_+\backslash G(\mathbb{Q}) \hookrightarrow G^{ad}(\mathbb{R})^+\backslash G^{ad}(\mathbb{R})$ is injective.
- $\Rightarrow$ $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K \to G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is a quasi-finite map, i. e. all fibers are finite.
- $\Rightarrow$ it is enough to show that $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite.

The finiteness of $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is an application of strong approximation, see [Mi03, p.55-59].

Step 2: The map $G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f) \to G(\mathbb{Q})\big\backslash X \times G(\mathbb{A}_f)$ is a bijection.
Consider the map

$$\begin{array}{ccc} G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f) & \longrightarrow & G(\mathbb{Q})\big\backslash X \times G(\mathbb{A}_f) \\ [x,a] & \longmapsto & [x,a] \end{array}$$

and let $[x,a] \in G(\mathbb{Q})\big\backslash X \times G(\mathbb{A}_f)$.

$\Rightarrow \ \exists\ q \in G(\mathbb{Q}),\ \exists\ x^+ \in X^+ :\ x = qx^+$, since $G(\mathbb{R})$ acts transitively on X and $G(\mathbb{Q}) \subset G(\mathbb{R})$ is dense by real approximation.

$\Rightarrow \ [x^+, q^{-1}a] = [qx^+, a] = [x, a]$ in $G(\mathbb{Q})\big\backslash X \times G(\mathbb{A}_f)$.

Hence, the map is surjective with $[x^+, q^{-1}a]$ being a preimage of $[x, a]$.

Now, let be $(x, a), (x', a') \in X^+ \times G(\mathbb{A}_f)$ with $[x, a] = [x', a']$ in the coset space $G(\mathbb{Q})\big\backslash X \times G(\mathbb{A}_f)$.

$\Rightarrow \ \exists\ q \in G(\mathbb{Q}) :\ (qx, qa) = (x', a')$ in $X \times G(\mathbb{A}_f)$.

$\Rightarrow \ q \in G(\mathbb{Q})_+$, since $qx = x'$ and $x, x' \in X^+$ implies that q does not interchange X^+ and X^-.

$\Rightarrow \ [x, a] = [x', a']$ in $G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f)$.

So, the map is injective.

This step implies that we may replace $G(\mathbb{Q})$ and X by $G(\mathbb{Q})_+$ and X^+, respectively.

Step 3: The canonical map and its surjectivity.
Consider the map

$$\begin{array}{ccc} \coprod_{g\in\mathcal{C}} \Gamma_g\backslash X^+ & \longrightarrow & G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f)\big/K \\ [x] & \longmapsto & [x, g] \end{array}$$

for $x \in X^+$ and $g \in \mathcal{C}$. Let $[x, a] \in G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f)\big/K$.

$\Rightarrow \ \exists\ g \in \mathcal{C},\ \exists\ q \in G(\mathbb{Q}_+),\ \exists\ k \in K :\ a = qgk$, since $\mathcal{C}$ is a set of representatives for the double cosets.

$\Rightarrow \ [x, a] = [q^{-1}x, gk] = [q^{-1}x, g]$.

$\Rightarrow \ [q^{-1}x] \in \Gamma_g\backslash X^+$ is a preimage of $[x, a] = [q^{-1}x, g]$.

So, the map is surjective.

Step 4: The injectivity on each connected component.
Consider the map

$$\begin{array}{ccc} \Gamma_g\backslash X^+ & \longrightarrow & G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f)\big/K \\ [x] & \longmapsto & [x, g] \end{array}$$

for some $g \in \mathcal{C}$ and let $[x], [x'] \in \Gamma_g\backslash X^+$ with $[x, g] = [x', g]$.

$\Rightarrow \ \exists\ q \in G(\mathbb{Q})_+,\ \exists\ k \in K :\ (qx, qgk) = (x', g)$ in $X^+ \times G(\mathbb{A}_f)$.

$\Rightarrow \ q = gk^{-1}g^{-1} \in gKg^{-1} \cap G(\mathbb{Q})_+ = \Gamma_g$.

$\Rightarrow \ [x] = [qx] = [x']$ in $\Gamma_g\backslash X^+$.

So, the map is injective for each $g \in \mathcal{C}$.

Step 5: Disjointness of the images of the components.
Let $[x] \in \Gamma_g\backslash X^+$ and $[x'] \in \Gamma_{g'}\backslash X^+$ for $g, g' \in \mathcal{C}$. Assume that $[x, g] = [x, g']$ in $G(\mathbb{Q})_+\big\backslash X^+ \times G(\mathbb{A}_f)\big/K$.

$\Rightarrow \ \exists\ q \in G(\mathbb{Q})_+,\ \exists\ k \in K : (qx, qgk) = (x', g')$, in particular $qgk = g'$.

$\Rightarrow \ [g] = [g']$ in $G(\mathbb{Q})_+\big\backslash G(\mathbb{A}_f)\big/K$.

$\Rightarrow \ g = g'$, since $g, g' \in \mathcal{C}$ are representatives of the double cosets.

So, if $g \neq g'$, then the images of $\Gamma_g\backslash X^+$ and $\Gamma_{g'}\backslash X^+$ are disjoint. In particular, the canonical map is injective.

This shows the bijectivity of the canonical map. □

Example 3.5. Let $B = M_2(\mathbb{Q})$ so that $G = \mathrm{GL}_{2,\mathbb{Q}}$, and let

$$K := K(N) := \prod_{\ell \text{ prime}} \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_\ell) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod \ell^{\mathrm{ord}_\ell(N)} \right\}$$

where $\mathrm{ord}_\ell(N)$ is the order of ℓ in N, i. e. it is the unique integer n such that $\ell^n | \, N$, but, $\ell^{n+1} \not| \, N$. Then

$$M_K(G,X) \simeq \coprod_{g\in\mathcal{C}} g\Gamma(N)g^{-1}\backslash X^+ \simeq \coprod_{g\in\mathcal{C}} \Gamma(N)\backslash\mathcal{H}$$

where $\Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\}$ is the standard modular subgroup of level N. Hence, $M_K(G,X)$ is the union of several copies of the (non-compact) modular curve $Y(N) = \Gamma(N)\backslash\mathcal{H}$, parameterizing elliptic curves with a symplectic level-N-structure.

Remark 3.6. If K is sufficiently small, then Γ_g is torsion-free and $M_K(G,X)$ will be an algebraic curve by a theorem of Bailey and Borel, see [Mi05, chapter 3].

If $K' \subset K$ are both sufficiently small, then the induced map

$$M_{K'}(G,X) \longrightarrow M_K(G,X)$$

is a regular map. Hence, we get a projective system of algebraic curves.

Definition 3.7 (The Shimura curve as projective system)**.** The projective system of algebraic curves

$$M(G,X) := \left\{ M_K(G,X) \right\}_K$$

where K runs through sufficiently small subgroups of $G(\mathbb{A}_f)$ is called the *Shimura curve* given by (G,X). Since (G,X) is uniquely determined by B/F, we call $M(G,X)$ also the *quaternionic Shimura curve* given by B.

Remark 3.8 (The $G(\mathbb{A}_f)$-action on the Shimura curve)**.** The group $G(\mathbb{A}_f)$ acts on $M(G,X)$ from the right by right multiplication. More precisely, an element $g \in G(\mathbb{A}_f)$ gives a map

$$M_K(G,X) \to M_{g^{-1}Kg}(G,X), \quad [x,a] \mapsto [x,ag]$$

for any K.

Remark 3.9 (Existence of canonical models)**.** Deligne has shown that the Shimura curve $M(G,X)$ can be defined over a number field, i. e. there is a common number field $E(G,X)$, such that all the curves $M_K(G,X)$ and all the maps $M_{K'}(G,X) \to M_K(G,X)$ are defined over the number field $E(G,X)$, see [De79].

Remark 3.10 (Moduli interpretation)**.** For $F = \mathbb{Q}$, the quaternionic Shimura curves $M(G, X)$ have a moduli interpretation. For $F \neq \mathbb{Q}$, no moduli interpretation exists, see [Mi03, chapter 5].

Some final words about the literature. The most accessible sources are the papers of Milne, [Mi03] and [Mi05]. Most things I learned from these two expositions. The standard source is the work of Deligne, [De71] and [De79]. But there, the topic is treated in much more generality.

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