Computing Approximations to Automorphic Forms

L J P Kilford
University of Bristol
l.kilford@gmail.com

Notation

We will let D be the quaternion algebra of discriminant 2; this is the Hamiltonian quaternions.

Let $\mathbf{A}_f$ be the finite adeles over $\mathbf{Q}$; we define $D_f = D \otimes_{\mathbf{Q}} \mathbf{A}_f$; this can be thought of as the restricted product over all primes l of $D \otimes \mathbf{Q}_l$; if $g \in D_f$ then the component g_p of g at the prime p can be viewed as an element of $M_2(\mathbf{Q}_p)$, for $p \neq 2$.

Let U be an open compact subgroup of $D_f^\times$. It is well known that one can write $D_f^\times$ as a *finite* union of disjoint double cosets, of the form

$$D_f^\times = \coprod_{i \in I} D^\times d_i U.$$

These can be computed efficiently, for instance by PARI programs first written by Daniel Jacobs.

If M is a positive integer, then we define $U_1(M)$ to be the open compact subgroup of $D_f^\times$ whose elements g have component $g_p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ at p with $c \equiv 0 \mod p$ and $d \equiv 1 \mod p$, and can be arbitrary at all other places. If G is a subgroup of $D^\times$ of finite index, we also define the open subgroup $U_1(M) \cdot G$ to be the subgroup of $U_1(M)$ whose image at primes dividing δ is in G.

If $\alpha \geq 1$, we define M_α to be the monoid consisting of 2 by 2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ over $\mathbf{Z}_p$ with nonzero determinant such that $p^\alpha | c$ and $p \nmid d$.

Let K be a complete subfield of $\mathbf{C}_p$ (the p-adic completion of $\mathbf{Q}_p$). Let L_k be the space of polynomials over K in one variable z of degree at most $k-2$. We will equip this with an action of M_α; let $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and let $h \in L_k$. Then we define the right action by

$$(h|_\gamma)(z) := (cz+d)^{k-2} \cdot h\left(\frac{az+b}{cz+d}\right).$$

Automorphic Forms

Let M be an odd positive integer. We define $S_k^D(U_1(M))$ to be the space of classical automorphic forms of level $U_1(M)$ and weight k for D, which is defined to be

$$\left\{ f : D_f^\times \to L_k : \; f(dgu) = f(g)u_p \text{ for all } d \in D^\times, \; u \in U_1(M) \right\}$$

Because we can write $D_f^\times$ as a finite union, we can determine $f \in S_k^D(U_1(M))$ by its values on the $\{d_i\}$; in other words, by the finite set $\{f(d_1), \ldots, f(d_2)\}$. We will use this later to perform calculations.

We also note that we can define Hecke operators on automorphic forms in terms of double cosets, in the same way that we define them on modular forms.

The Jacquet-Langlands Correspondence

Here is an explicit version of the correspondence.

Theorem 1 (Jacquet-Langlands, Arthur) *Let $k \geq 3$ be an integer and let $N \geq 5$ be a positive integer. There is an isomorphism between the space of modular forms $S_k(\Gamma_1(N) \cap \Gamma_0(2))$ and the space of automorphic forms $S_k^D(U_1(N))$. If $k = 2$ then the space of modular forms is isomorphic to the quotient of the space of automorphic forms by the norm form.*

We note that this correspondence respects the action of the Hecke operators; in other words, the Hecke operator T_p has the same eigenvalues for corresponding automorphic forms and modular forms.

Weight 1

This statement of Jacquet-Langlands leaves a natural question open — what happens in weight 1? Firstly, we need to work out what is going on on the automorphic side. The classical definition involves polynomials of degree at most $k-2$; how do we make sense of polynomials of degree -1?

An insight of Buzzard is to consider p-adically convergent power series in one variable; polynomials form a subspace of these, and the classical theory generalizes to this much larger situation.

Families of modular forms

We can define collections of modular forms called *families*. Let g be a particular modular form of weight k and let p be a prime number. We say that the set g_i of modular forms forms a p-adic family if the weight of g_i is $k + (p-1) \cdot p^i$, and the Fourier expansion of each of the g_i satisfies

$$g(q) \equiv g_i(q) \mod p^i,$$

where we say that two Fourier series are congruent modulo p^n if the Fourier coefficients of their difference all vanish modulo p^n.

The canonical example of families of modular forms is given by Eisenstein series;

We can define similar families of automorphic forms of weight $k + (p-1) \cdot p^i$, related by their eigenvalues for the T_p.

Convergent power series

In order to deal with weight 1, we will have to extend our idea of what automorphic forms are. We first introduce convergent power series.

Let K be a complete subfield of $\mathbf{C}_p$. Then we define $A_{k,1}$ to be the ring $K\langle z \rangle$ of power series $\sum_{n \in \mathbf{N}} a_n z^n$ such that $a_n \to \infty$ as $n \to \infty$. We call these *convergent power series*.

We see that the right action by elements of M_α on $A_{k,1}$, extending the right action of M_α on L_k,

$$(h|_\gamma)(z) := (cz+d)^{k-2} \cdot h\left(\frac{az+b}{cz+d}\right)$$

does send $A_{k,1}$ to itself, so this action is well-defined.

Overconvergent automorphic forms

We can now define *overconvergent automorphic forms*.

Let M be a positive integer which is prime to δ. We define $S_k^{D,\dagger}(U_1(M))$ to be the space of *overconvergent automorphic form s* of level $U_1(M)$ and weight k for D, which is

$$\left\{ f : D_f^\times \to A_{k,1} : \; f(dgu) = f(g)u_p \text{ for all } d \in D^\times, \; u \in U_1(M) \right\}.$$

We can define overconvergent automorphic forms for other open compact subgroups in a similar way.

We see that these spaces are infinite-dimensional, and also that they contain the spaces of classical automorphic forms, because a polynomial is certa inly a convergent power series. On the other hand, we can determine an overconvergent form f by its values on the $\{d_i\}$; in other words, by the tuple of convergent power series $\{f(d_1), \ldots, f(d_n)\}$.

We also note that the action of the Hecke operators T_l and U_p is well-defined; we use the same definitions as we used in the classical case, ext ending the weight k action from polynomials to convergent power series.

Overconvergent automorphic forms versus overconvergent modular forms

It is emphasized in the introduction to Buzzard's paper on familes that one advantage of considering overconvergence for automorphic forms is that the ge ometric aspects of the definition are much simpler than those in the definition of overconvergent modular forms; we simply change the base ring to def ine overconvergent automorphic forms, whereas we need to set up a lot of machinery to define overconvergent modular forms.

Checking computations

If we are computing spaces of classical automorphic forms p-adically, then because these are finite-dimensional we can actually compute the action of Hecke operators on them exactly, because we know that their characteristic power series have integral coefficients of bounded size. We compute them to a sufficiently high p-adic precision, and then we can rewrite these p-adic numbers as rational integers.

Let us consider some numerical examples. Firstly, we compute the Hecke polynomial of T_3 acting on the space $S_5^D(U_1(7))$ of classical automorphic forms; it is

$$(x^4 + 288x^2 + 20448) \cdot (x^4 + 18x^3 + 39x^2 - 1242x + 4761).$$

It can be verified that the Hecke polynomial of T_3 acting on the space of classical modular forms $S_5^{2-\text{new}}(\Gamma_1(7) \cap \Gamma_0(2))$ is the same as this.

Using MAGMA, we can compute the Hecke polynomial of U_{11} acting on the spaces $S_3^{2-\text{new}}(\Gamma_1(11) \cap \Gamma_0(2))$ and $S_3^{4-\text{new}}(\Gamma_1(11)) \cap \Gamma_0(4))$ of classical modular forms, and we see that these polynomials are $x^2 - 14x + 121$ and $x^2 + 22x + 121$, respectively. Using PARI, we see that the Hecke polynomial of U_{11} on the space of automorphic forms $S_3^D((1+) \cdot U_1(11))$ is given by

$$(x^2 - 14x + 121) \cdot (x^2 + 22x + 121)^2,$$

as predicted (we see the part which comes from 4-new forms appearing with multiplicity two here).

We can also compute automorphic forms with more restrictive level structure at 2. The characteristic polynomial of the Hecke operator U_5 acting on the space of automorphic forms of weight 2 and level $\Gamma_0(5)$ with level structure $1 + \mathfrak{m}^3$ at 2 is given by

$$(x^2 - 4x + 5)^4 \cdot (x^2 + 2x + 5)^2 \cdot (x^2 - 2x + 5)^3 \cdot (x-1)^3 \cdot (x+1)^2 \cdot (x-5),$$

where the $x - 5$ comes from the norm form (because $k = 2$), the $(x-1)^3$ and the $x^2 + 2x + 5$ factors come from classical modular forms of level $\Gamma_0(40)$, the $(x+1)^2$ from modular forms of level $\Gamma_0(20)$, and the quadratic factors $(x^2 - 4x + 5)^4$ and $(x^2 - 2x + 5)^3$ come from classical modular forms of level $\Gamma_0(80)$.

One important feature of these calculations is that they give us an independent way to check that the modular forms algorithms in MAGMA are giving the correct answers. Those algorithms use the theory of modular symbols and were programmed independently to the current work, so because we are using two different algebra packages and two different sets of programs, and still getting the same answer, we can be more certain that our programs are giving the correct results.