Fundamental domains and Noncongruence subgroups

19th August 2008

Non Congruence subgroups: don't contain $\Gamma(N)$ for any N.

Big problem: understand modular forms for these groups, e.g., Atkin Swinnerton-Dyer congruences (much done by Scholl).

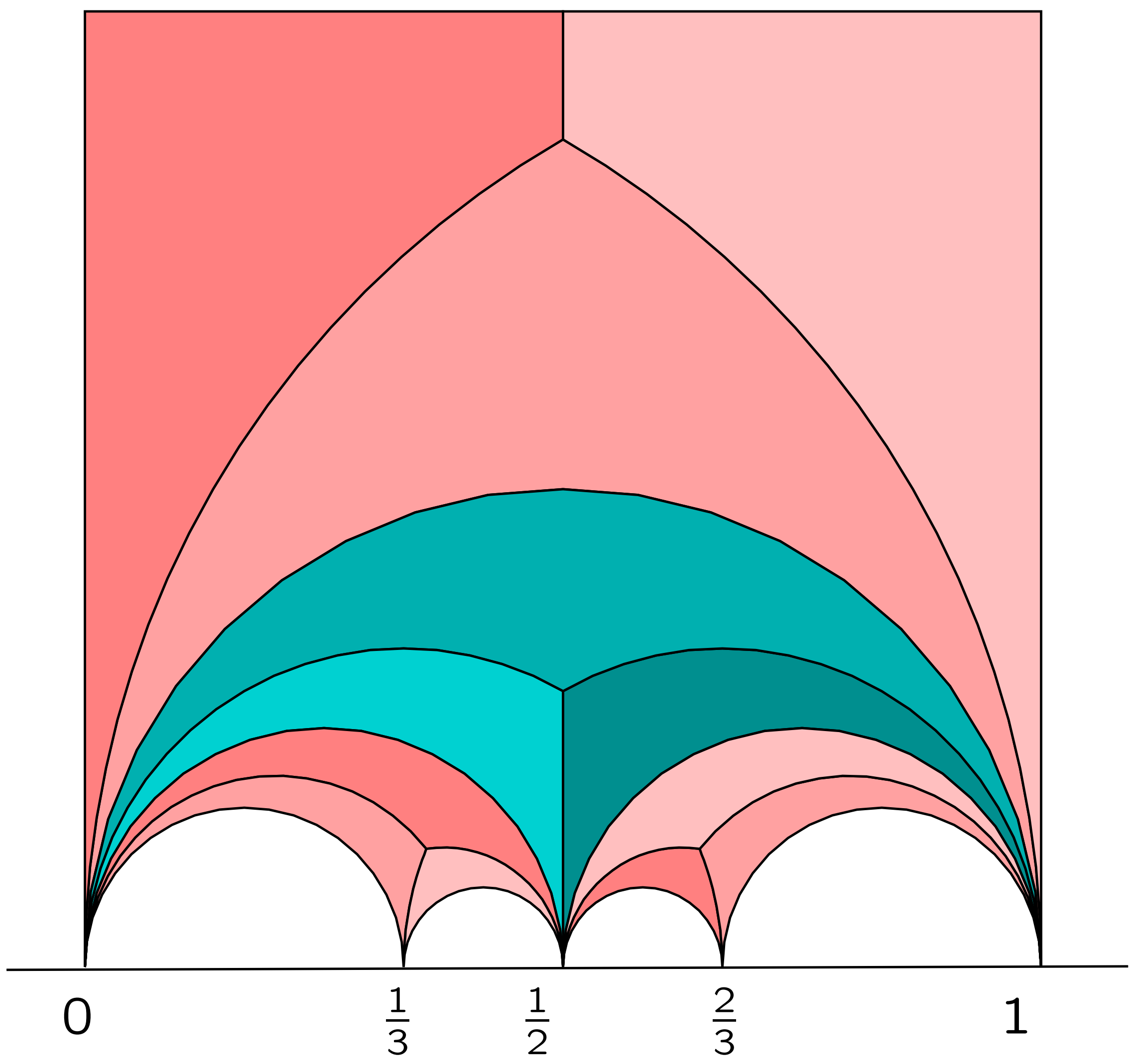
Smaller problem: Determine whether a subgroup is non-congruence.

This talk is about the small problem. Based on work of Lim, Lang, Tan; Kulkarni; Wohlfahrt. New: groups not containing $-I$.

This is beginning of proposed joint work with Ian Kiming and Mathias Schüt. Follow on from joint work with Bill Hoffman

Kulkarni gives an algorithm for finding
nice fundamental domains for subgroups of $SL_2(Z)$. Domain is more or less union of domains for $\Gamma_0(2)$. Gives minimal number of generators.

Example: Fundamental domain for $\Gamma_0(6)$:



Aside note:

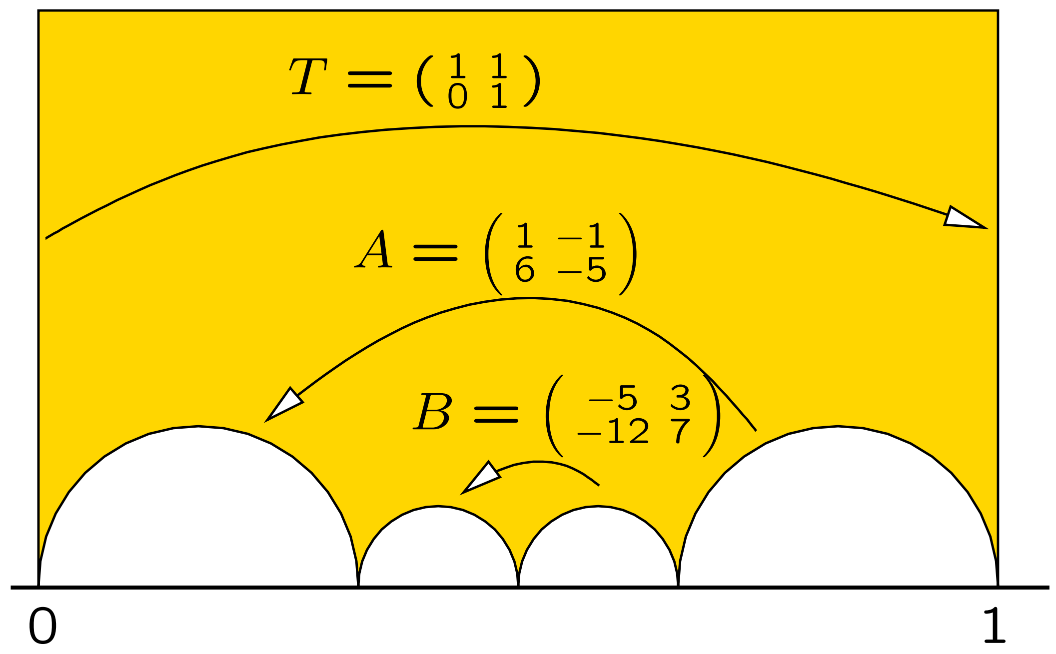
Picture (not including text) produced by pari functions making xfig output

```
r = [0,1;-1,1]; t=[1,1;0,1]; i=[1,0;0,1]; s=[0,-1;1,0];
m=[1,0;1,1]*s/[1,0;1,1];
tri = ["oo",0,1/2+sqrt(3)/2*I];
makeFilledImageOfPolygon("gamma0_6.fig",tri,[i,r,r^2],[27,28,29],1,1)
makeFilledImageOfPolygon("gamma0_6.fig",tri,[m*i,m*r,m*r^2],[15,16,17],1,0)
makeFilledImageOfPolygon("gamma0_6.fig",tri,[m*t*i,m*t*r,m*t*r^2],[27,28,29],1,0)
makeFilledImageOfPolygon("gamma0_6.fig",tri,[m/t*i,m/t*r,m/t*r^2],[27,28,29],1,0)
```

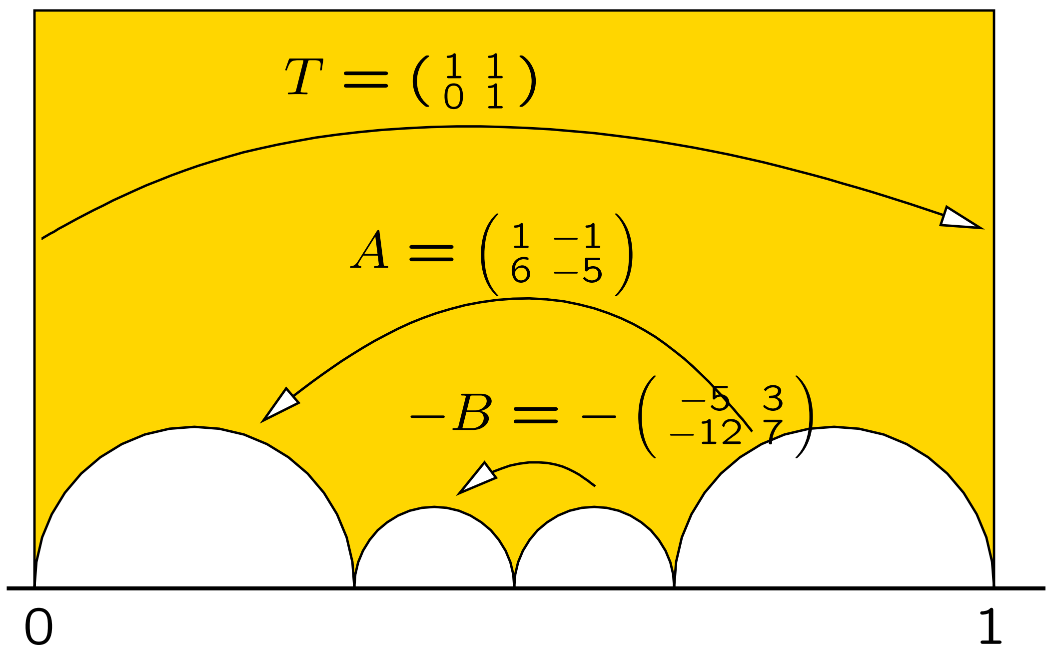
Next page:

```
fd=["oo",0,1/3,1/2,2/3,1];
makeFilledImageOfPolygon("gamma0_6a.fig",fd,[i],[31],0.6,1)
```

Domain for the congruence subgroup $\Gamma_1(6)$:



Domain for the noncongruence subgroup $\langle T, A, -B \rangle$



How to determine whether a subgroup Γ of $SL_2(\mathbf{Z})$ is a congruence subgroup:

Definition (Wohlfahrt): The *general level* of Γ is the lmc of the cusp widths.

Theorem (following Wohlfahrt): If Γ is a congruence subgroup of general level N, then it's level as a congruence subgroup is N or $2N$.

Proof uses the group ${}_N\Gamma$, generated by the parabolic matrices in $\Gamma(N)$.

Algorithm: If Γ has general level N, test whether the generators of $\Gamma(N)$ or $\Gamma(2N)$ are in Γ. Use a (modified) algorithm of Lim, Lang, Tan, based on Kulkarni's results. (or an alternative algorithm) (both programmed in gap).

Algorithm to test whether a matrix m is in Γ
(Lim, Lang, Tan) (up to $\pm I$):

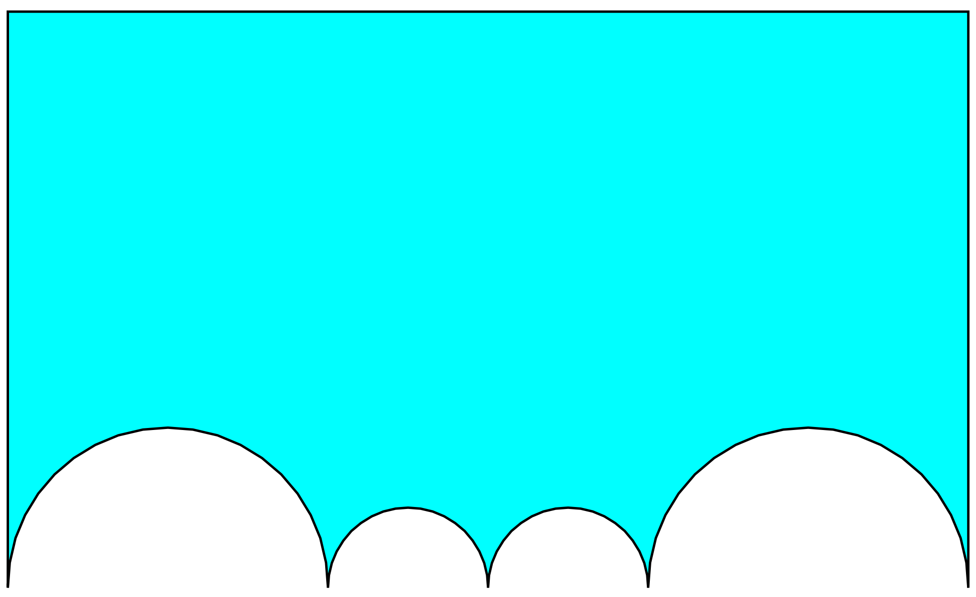
- Apply m to the choosen domain $\mathcal{F}$ for Γ.
- Multiply by generators of Γ to approach $\mathcal{F}$.
- Iff you get to a region unequal to but overlapping $\mathcal{F}$, then $m \notin \Gamma$.

Example for $\Gamma_0(11)$ and $m = \left(\begin{smallmatrix} 7 & 1 \\ 6 & 1 \end{smallmatrix}\right)$

Gap code:

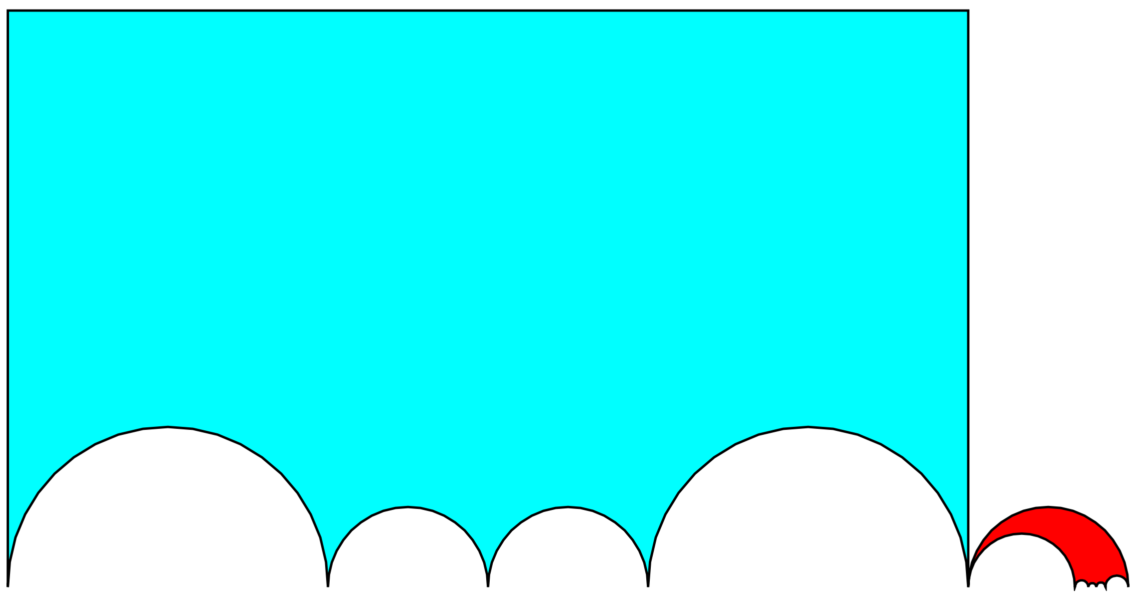
```
LoadPackage("congruence");  # by Ann Dooms, Eric Jespers, Alexander Konovalov

G11:=Gamma0(11);
FS11:=FareySymbol(G11);
gens:=GeneratorsByFareySymbol(FS11);
```

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 7 & -2 \\ 11 & -3 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 8 & -3 \\ 11 & -4 \end{pmatrix},$$



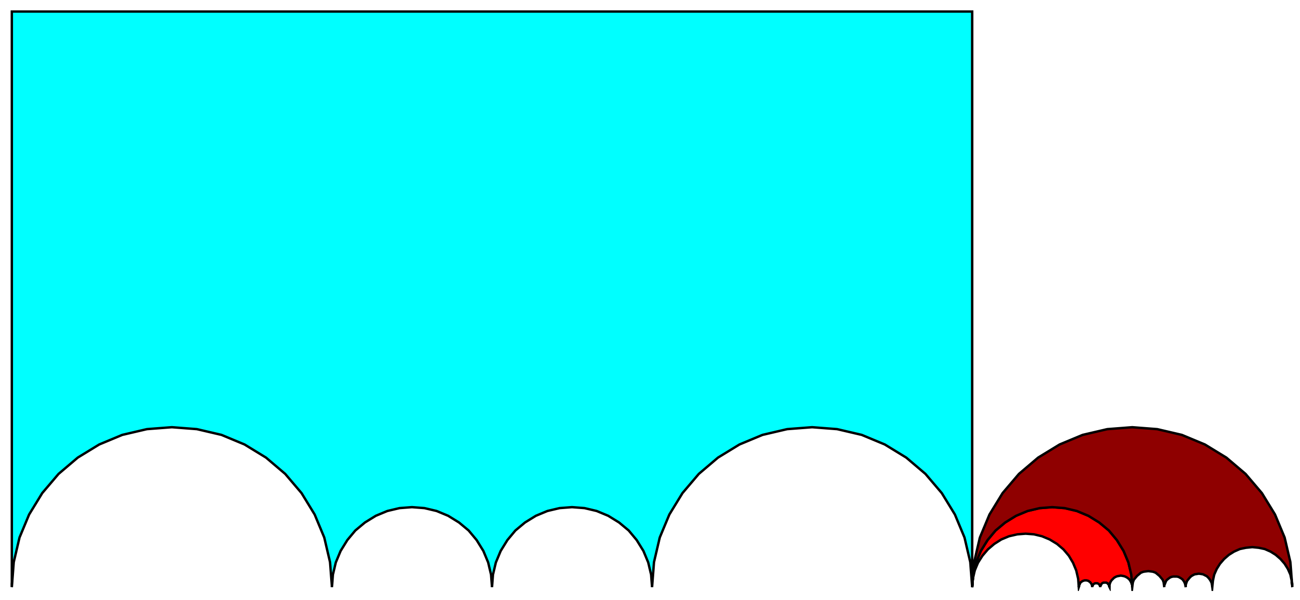
gap code:

```
Read("membershiptest2.g");
g:= [[7,1],[6,1]];
glue_list:=gluing_matrices(FS11);
w:=find_word_ver2(FS11,glue_list,g);

[1,-3,2]
not in group

# alternative algorithm:

Read("membershiptest1.g");
w:=find_word(FS11,glue_list,g);

[1,-2,3]
not in group
```



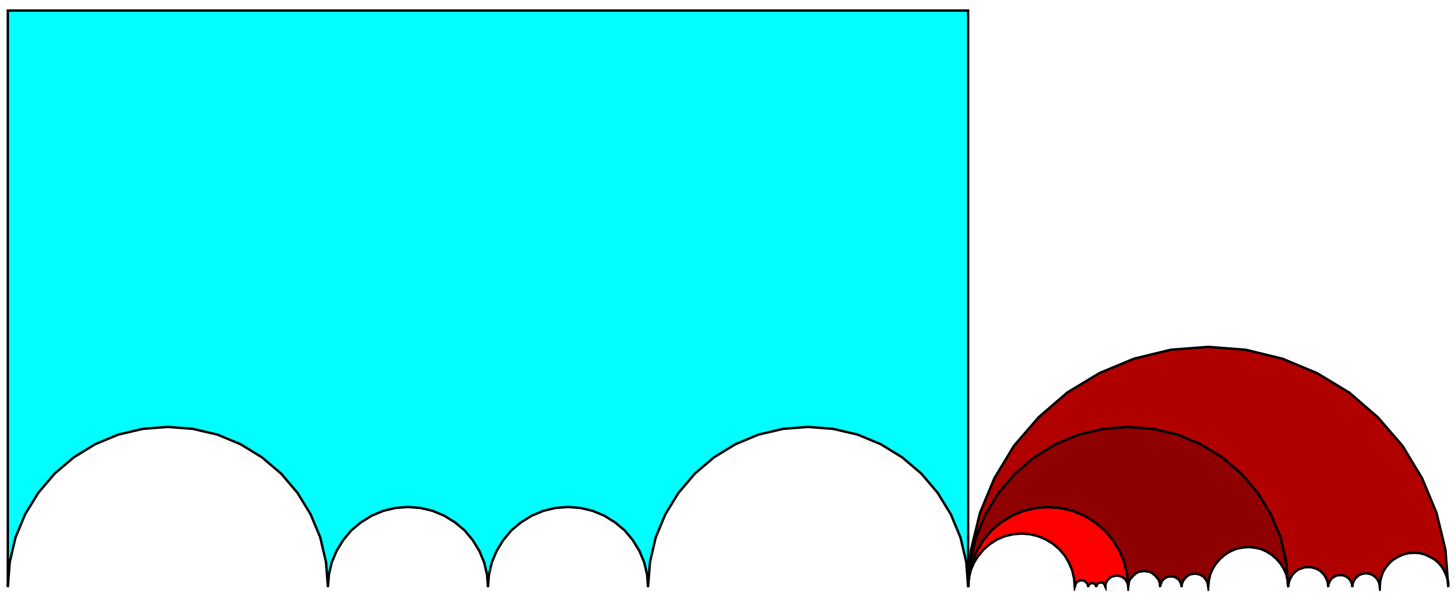
pari:

```
makeFilledImageOfPolygon("gamma0_11a.fig",fd,[i,[7,1;6,1]],[3,4],0.6,1)
```



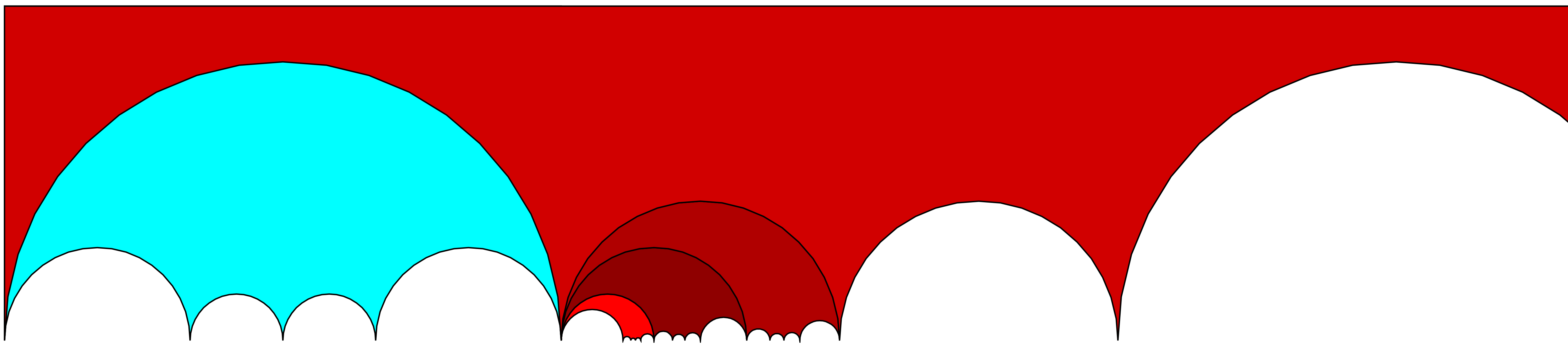
pari:

```
makeFilledImageOfPolygon("gamma0_11a.fig",fd,[[7,1;6,1]*g1^(-1)],[18],0.6,0)
```



pari:

```
makeFilledImageOfPolygon("gamma0_11a.fig",fd,[[7,1;6,1]*g1^(-1)*g3],[19],0.6,0)
```

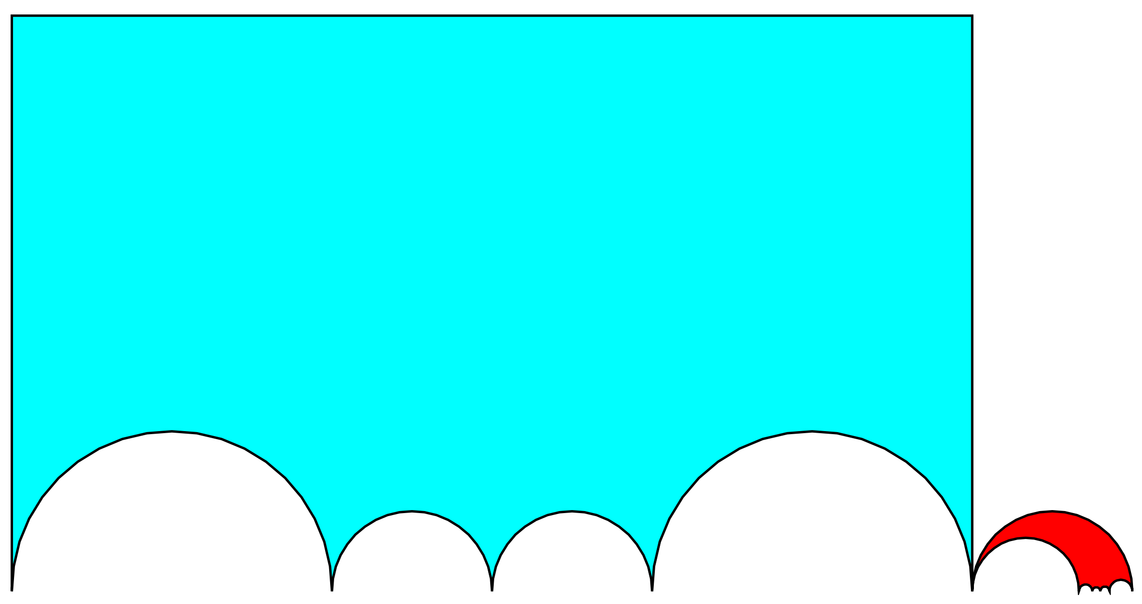


pari:

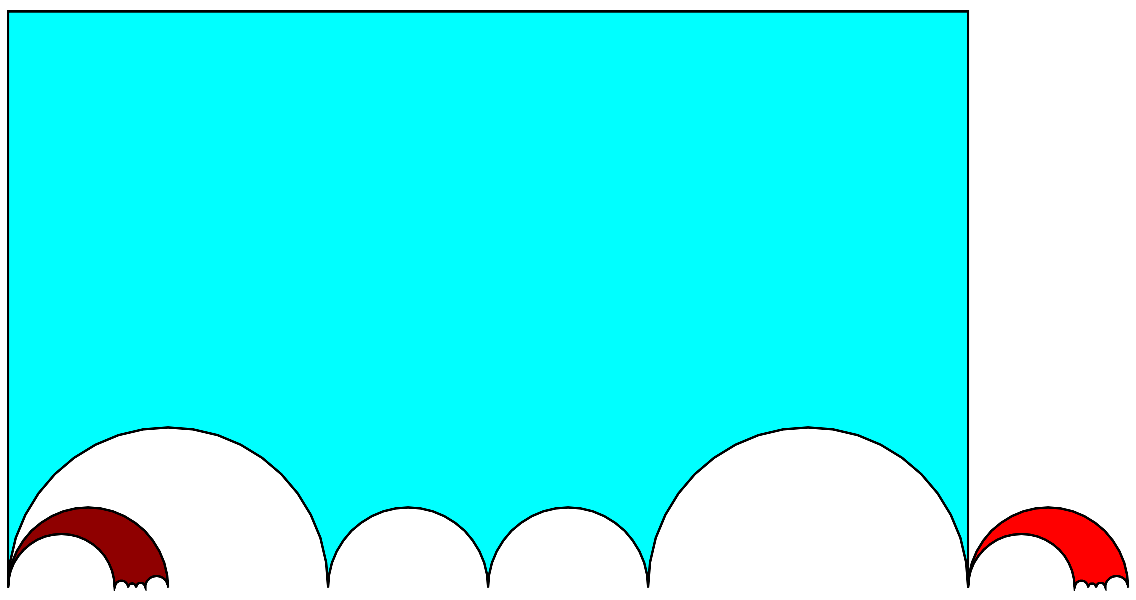
```
makeFilledImageOfPolygon("gamma0_11a.fig",fd,[[7,1;6,1]*g1^(-1)*g3*g2^(-1)],[20],0.6,0)
```

Algorithm to test whether a matrix is in Γ
(like method to show T and S generate $\mathrm{SL}_2(\mathbf{Z})$)
(being investigate by Cristian Caranicia)

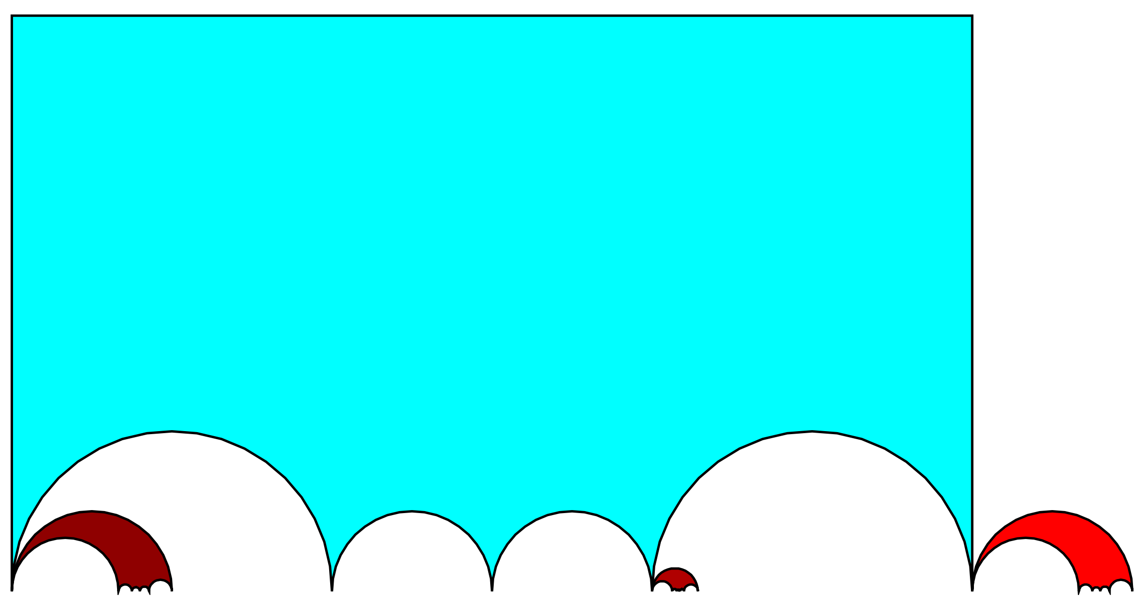
- Apply m to the choosen domain $\mathcal{F}$ for Γ.
- Multiply by generators of Γ to approach $\mathcal{F}$.
- Iff you get to a region unequal to but overlapping $\mathcal{F}$, then $m \notin \Gamma$.



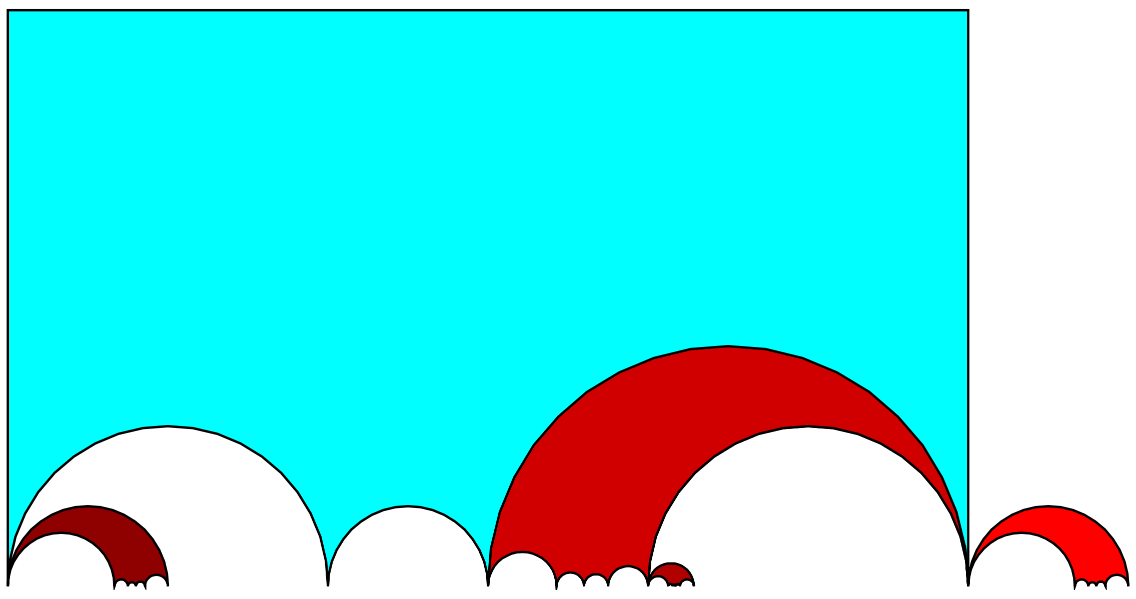
```
makeFilledImageOfPolygon("gamma0_11a.fig",fd,[i,[7,1;6,1]],[3,4],0.6,1)
```



```
makeFilledImageOfPolygon("gamma0_11e.fig",fd,[i,[7,1;6,1],g1^(-1)*[7,1;6,1]],[3,4,18],0.6,1)
```



makeFilledImageOfPolygon("gamma0_11e.fig",fd,[g2*g1^(-1)*[7,1;6,1]],[19],0.6,0)



makeFilledImageOfPolygon("gamma0_11e.fig",fd,[g3^(-1)*g2*g1^(-1)*[7,1;6,1]],[20],0.6,0)

Back to testing whether a group is congruence

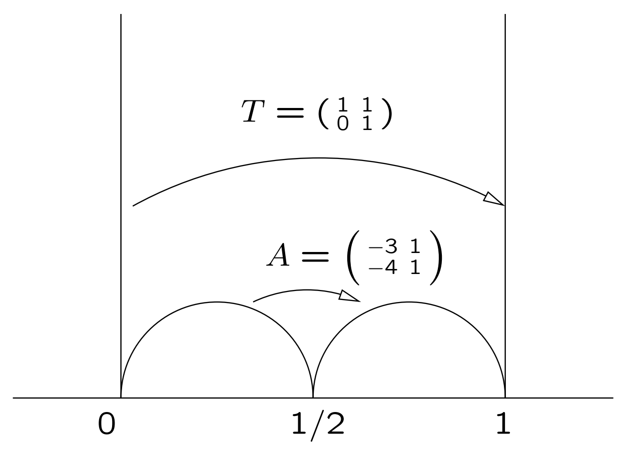
If general level of Γ is N, need to test whether $\Gamma(N)$ or $\Gamma(2N)$ is in Γ.

If this is true projectively, we just have a problem of signs.

Questions:
- How many groups Γ with $I \notin \Gamma$ and $\Gamma = \Gamma_0(N)$ are noncongruence?
- How do their forms (esp. odd wt) compare?
- Do ASWD relations hold?

(these are motivating questions; no answers today) example: $\Gamma = \Gamma_1(4)$.

Domain and generators:



Groups with the same domain $\mathcal{F}$
$\langle T, A \rangle = \Gamma_1(4)$
$\langle T, A, -I \rangle = \Gamma_0(4)$
$\langle T, -A \rangle$
$\langle -T, A \rangle$
$\langle -T, -A \rangle$

Table of cusps and stabilizers, where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$:

cusp	stabilzer	width	regular in...?			
			$\langle T, A\rangle$	$\langle T, -A\rangle$	$\langle -T, A\rangle$	$\langle -T, -A\rangle$
∞	T	1	Y	Y	n	n
0	$AT = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = ST^{-4}S^{-1}$	4	Y	n	n	Y
1/2	$A = -BT^{-1}B^{-1}$	1	n	Y	n	Y
		dim $S_5(\Gamma)$	1	1	2	1
		level as congruence subgroup	4	8	8	4

Generators for $\Gamma(4)$ in terms of A and T.

		Is the generator contained in...			
		$\langle T, A\rangle$	$\langle T, -A\rangle$	$\langle -T, A\rangle$	$\langle -T, -A\rangle$?
$\begin{pmatrix} -7 & 4 \\ -16 & 9 \end{pmatrix}$	$= A^4$	y	y	y	y
$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$	$= AT$	y	n	n	y
$\begin{pmatrix} -11 & 4 \\ -36 & 13 \end{pmatrix}$	$= AT^{-1}A^{-2}$	y	n	n	y
$\begin{pmatrix} -11 & 8 \\ -40 & 29 \end{pmatrix}$	$= AT^{-2}A$	y	y	y	y
$\begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$	$= TA$	y	n	n	y

This shows that $\langle T, A\rangle$ and $\langle -T, -A\rangle$ are congruence subgroups of level 4.

$\Gamma(8)$ can be generated by 33 matrices. Writing these in terms of A and T, each generator is given as a product of an even number of As and Ts, so $\langle -T, A\rangle$ and $\langle T, -A\rangle$ are congruence subgroups of level 8.

Example: $\Gamma_1(6)$ generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & -1 \\ 6 & -5 \end{pmatrix}, B = \begin{pmatrix} -5 & 3 \\ -12 & 7 \end{pmatrix}$$

8 subgroups with same fundamental domain, not containing $-I$.

$\Gamma(6)$ can be generated by the following matrices

$$\begin{aligned}
\begin{pmatrix} -11 & 6 \\ -24 & 13 \end{pmatrix} &= B^2 \\
\begin{pmatrix} -17 & 6 \\ -54 & 19 \end{pmatrix} &= (AB^{-1})^3 \\
\begin{pmatrix} -29 & 18 \\ -108 & 67 \end{pmatrix} &= AB^{-1}A^{-1}B \\
\begin{pmatrix} 19 & -12 \\ 84 & -53 \end{pmatrix} &= ABA^{-1}B \\
\begin{pmatrix} -17 & 12 \\ -78 & 55 \end{pmatrix} &= A^2B^{-1}A \\
\begin{pmatrix} -41 & 30 \\ -108 & 79 \end{pmatrix} &= BA^{-1}B^{-1}A \\
\begin{pmatrix} -23 & 18 \\ -78 & 61 \end{pmatrix} &= AB^{-1}A^2
\end{aligned}
\qquad
\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} &= AT \\
\begin{pmatrix} 31 & -12 \\ 168 & -65 \end{pmatrix} &= AT^{-1}AB^{-1} \\
\begin{pmatrix} -29 & 12 \\ -162 & 67 \end{pmatrix} &= AT^{-2}B^{-1} \\
\begin{pmatrix} -71 & 30 \\ -258 & 109 \end{pmatrix} &= AB^{-1}TB^{-1} \\
\begin{pmatrix} 37 & -30 \\ 132 & -107 \end{pmatrix} &= AB^{-1}T^{-1}A \\
\begin{pmatrix} 7 & -6 \\ 6 & -5 \end{pmatrix} &= TA
\end{aligned}$$

up to parity and reordering, only need to check AT, AB (implies BT is in group) so only $\langle A, B, T \rangle$ and $\langle -A, -B, -T \rangle$ are congruence level 6.

cusp	stabilzer	width	regular in ...?			
			$\langle T, A, B\rangle$	$\langle T, -A, B\rangle$	$\langle -T, A, B\rangle$	$\langle -T, -A, B\rangle$
∞	T	1	Y	Y	n	n
0	$AT = (T^{-6})^S$	6	Y	n	n	Y
$\frac{1}{2}$	B	3	Y	Y	Y	Y
$\frac{1}{3}$	$AB^{-1} = (T^2)^{\left(\begin{smallmatrix}1 & 0\\ 3 & 1\end{smallmatrix}\right)}$	2	Y	n	Y	n
		$\dim S_3(\Gamma)$	0	1	1	1
		level if congruence	6	12	–	–

cusp	regular in ...?			
	$\langle T, A, -B\rangle$	$\langle T, -A, -B\rangle$	$\langle -T, A, -B\rangle$	$\langle -T, -A, -B\rangle$
∞	Y	Y	n	n
0	Y	n	n	Y
$\frac{1}{2}$	n	n	n	n
$\frac{1}{3}$	n	Y	n	Y
$\dim S_3(\Gamma)$	1	1	2	1
level if congruence	–	–	12	6

Γ(12) can be generated by 97 matrices

Up to parity and rearanging the letters A, B, T only need to check TB e.g.,

$$TB^{-1}A^{-2} = \begin{pmatrix} 169 & -36 \\ 108 & -23 \end{pmatrix}$$

So the groups $\langle T, A, B\rangle, \langle T, -A, B\rangle, \langle -T, A, -B\rangle, \langle -T, -A, -B\rangle$, all contain Γ(12). The remaining groups are noncongruence subgroups.