Computation of Maass waveforms

Fredrik Strömberg
TU Darmstadt, Germany

Background

1 Definition

Consider the hyperbolic upper half-plane $\mathbf{H} = \{x+iy \in \mathbb{C} \,|\, y>0\}$ equipped with the metric $ds^2 = y^{-2}\left(dx^2+dy^2\right)$ and measure $d\mu = y^{-2}dxdy$. Let Γ be a co-finite, non co-compact Fuchsian group, i.e. Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ such that the quotient $\mathscr{M} = \Gamma\backslash\mathbf{H}$ has finite hyperbolic area but is not compact (it has at least one cusp).
A Maass waveform (cusp form) is a square-integrable real-analytic function on $\Gamma\backslash\mathbf{H}$ which is an eigenfunction of the Laplace-Beltrami operator $\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$. I.e. $\phi \in C^2(\mathbf{H})$ is said to be a *Maass waveform* if it satisfies the following conditions:

$$(\Delta+\lambda)\,\phi(z) = 0, \lambda = \frac{1}{4}+R^2 > 0, \qquad (1)$$

$$\phi(\gamma z) = \phi(z), \forall \gamma \in \Gamma, \forall z \in \mathbf{H}, \qquad (2)$$

$$\int_{\Gamma\backslash\mathbf{H}} |\phi|^2 d\mu < \infty. \qquad (3)$$

It is known that for congruence subgroups a function ϕ satisfying 1.-3. is automatically cuspidal, i.e. $\phi(x+iy) \to 0$ as $y \to \infty$ and has a Fourier expansion

$$\phi(z) = \sum_{n \neq 0} c_n \kappa_n(y)\, e(nx)$$

where $\kappa_n(y) = \sqrt{y}K_{iR}(2\pi|n|y)$ and $e(x) = e^{2\pi i x}$.

2 Motivation

The Laplace-Beltrami operator Δ can be interpreted as a stationary Schrödinger operator and Maass waveforms correspond to bound quantum eigenstates on $\mathscr{M}$. The semi-classical limit ($\hbar \to 0$) corresponds to $\lambda \to \infty$. Since the classical billiard system on $\mathscr{M}$ is strongly chaotic, the properties of Maass waveforms and their eigenvalue distribution for large λ are objects of interest in the study of so-called „quantum chaos“.
Although generic Maass waveforms are (seemingly) transcendental in their nature, there are special cases which also has a more number theoretical interpretation. These special cases, CM-forms, exists only on congruence subgroups together with non-trivial characters and both their spectral parameters and Fourier coefficients can be explicitly computed.

3 Some generalizations of Maass waveforms

If $\chi : \Gamma \to \mathbb{C}$ is a character we can replace (2) by

$$\phi(\gamma z) = \chi(\gamma)\,\phi(z) \qquad (2')$$

and in general, if $k \in \mathbb{R}$ and $v : \overline{\Gamma} \to \mathbb{C}$ is a multiplier system of weight k for $\overline{\Gamma}$ (here $\overline{\Gamma}$ is the inverse image of the natural projection of Γ in $\mathrm{SL}_2(\mathbb{R})$) we can replace (2) by

$$\phi(\gamma z) = j_\gamma(z)^k v(\gamma)\,\phi(z) \qquad (2'')$$

and (1) by

$$(\Delta_k+\lambda)\,\phi(z) = 0, \lambda = \frac{1}{4}+R^2 > 0 \qquad (1'')$$

where $j_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)}(z) = e^{i\mathrm{Arg}(cz+d)}$ and $\Delta_k = \Delta - iyk\frac{\partial}{\partial x}$.
We can also loosen up the condition (3) which implies some type of growth-bounds at infinity (i.e. cuspidality of ϕ for congruence Γ if $\lambda > 0$). The first step would be to allow for polynomial growth at infinity. This is achieved by for example *Eisenstein series* $E(z;s)$ which belongs to the continuous spectrum of Δ. Here

$$E(z;s) = y^s + \varphi(s)\,y^{1-s} + O\left(e^{-\varepsilon y}\right), \quad \text{as}\, y \to 0$$

for some $\varepsilon > 0$. We can also allow exponential growth. Let $P_\phi(z)$ be a polynomial in $q = e(z) = e^{2\pi i z}$. If ϕ satisfies (1) and (2) and

$$\phi(z) = P_\phi(z) + O\left(e^{-\varepsilon y}\right), \quad \text{as}\, y \to \infty \qquad (3''')$$

for some $\varepsilon > 0$ we say that ϕ is a *weak Maass waveform*.

Algorithm

4 The core algorithm consists of four steps

1. Rapid convergence of Fourier series$\Rightarrow$ truncation at $M_0 = M(Y_0)$ and $\phi \approx \hat{\phi}$ for $Y > Y_0$ with
$$\hat{\phi}(z) = \sum_{|n| \leq M_0} c_n \kappa_n(y)\, e(nx)$$
2. Fourier inversion over $z_m = x_m + iY$, $1-Q < m < Q$ with $Q > M_0$:
$$c_n \kappa_n(Y) = \sum_{m=1-Q}^{Q} \hat{\phi}(z_m)\, e(-nx_m)$$
3. Automorphy of ϕ: $\phi(\gamma z) = v(\gamma)\,\phi(z) \Rightarrow \hat{\phi}(z_m^*) \approx \hat{\phi}(z_m)$:
$$c_n \kappa_n(Y) \approx \sum_{m=1-Q}^{Q} \hat{\phi}(z_m^*)\, e(-nx_m) = \sum_{|l| \leq M_0} V_{nl} c_l$$
where z_m^* is the *pullback* of z_m to the fundamental domain of Γ.
4. Solve the resulting homogeneous system for the coefficients $\vec{c} = \vec{c}(Y,R) \in \mathbb{C}^{2M_0+1}$ using suitable normalization e.g. $c(1) = 1$ to obtain a Hecke normalized newform.

5 Phase 1 (locating eigenvalues)

For an arbitrary R, use two different Y's, Y_1 and Y_2 and compute $\vec{c} = \vec{c}(Y_1,R)$ and $\vec{c}' = \vec{c}(Y_2,R)$. If $\lambda = \frac{1}{4}+R^2$ really is an eigenvalue of Δ then these vectors should be identical (up to some given error). Locating eigenvalues can thus be done by finding simultaneous zeros of a set:

$$h_j(R) = c(i_j) - c'(i_j),\; j = 1,2,3$$

where for example $i_1 = 2$, $i_2 = 3$ and $i_3 = 4$.

6 Phase 2

Compute more coefficients using "phase 2":

$$c(n) = \frac{\sum_{|n| \leq M_0} V_{nl} c(l)}{\kappa_n(Y)},$$

where n is allowed to be greater than M_0, using successively decreasing Y and increasing Q.

Implementation

7 The program

The original implementation of this algorithm in the setting of $PSL_2(\mathbb{Z})$ was made by Dennis Hejhal in the late 80's and beginning 90's using FORTRAN77. The current version is implemented in Fortran 90/95 and consist of a package of several Fortran 90/95 modules and programs interfacing these modules. To work with Maass waveforms, download the file *maasswf.tar.gz*, unzip/tar it and follow the instructions in the readme file. You will then have the program *maasswf* which can be used to locate eigenvalues (Phase 1) and compute more coefficients (Phase 2). There is also functionality to produce indata to *lcalc* if you wish to use the coefficients to compute L-functions and to produce data files data files which can be plotted with for example SAGE and pylab. The current version of *maasswf* is limited to $\Gamma = \Gamma_0(N)$ and real characters. Further versions will extend this functionality. There is also an ongoing project to make the Maass waveform programs available through SAGE.

Contact: Fredrik Strömberg
Fachbereich Mathematik, AG AGF, TU Darmstadt, Schloßgartenstraße 7
64289 Darmstadt, Germany.
Email: stroemberg@mathematik.tu-darmstadt.de.
Web: http://www.mathematik.tu-darmstadt.de/~stroemberg

8 Find eigenvalues

First we need to find an eigenvalue. Consider $N = 5$ and trivial character. We do that by running a search algorithm:

```
>./maasswf -find 1 -lvl 5 -ch 1 -Rs 0 -Rf 7
   5  0.000     1     3.2642513026365152      1 -1   1.1147E-13
   5  0.000     1     4.8937812914384189      1 -1   5.7732E-14
   5  0.000     1     4.8937812914384446      1  1   1.2079E-13
   5  0.000     1     6.2149037377076759      1  1   3.2863E-14
   5  0.000     1     6.2149037377076901      1 -1   1.7764E-14
   5  0.000     1     6.5285026052730224      0  1   8.1624E-13
```

The output is given as a list where each row is
$N \quad k \quad \chi \quad R \quad \mu_0 \quad \mu_1, \cdots, \mu_\kappa, \quad \varepsilon$
where N is the level, k is the weight (here 0) χ is the character (here $\chi_5 = \left(\frac{\cdot}{5}\right)$ denoted by 1), R is the spectral parameter, $\mu_0 = 0$ if the corresponding function is even with respect to $J : z \mapsto -\bar{z}$ and $\mu_0 = 1$ if it is odd. Then follows a list $\mu_1, \ldots, \mu_\kappa$ consisting of eigenvalues of Atkin-Lehner involutions. The last entry is an error estimate. By adding the option

```
-o testlist.txt
```

we can print the list directly to the file *testlist.txt*.

9 Obtain more coefficients

Suppose that we want to compute a longer list of Fourier coefficients for the last form in the list above and that we want to use these coefficients in *lcalc* to compute, for example, zeros of the corresponding L-function. With the following command:

```
./maasswf -start 6 -stop 6 -c 10000 -f testlist.txt  -lcalc 1
```

we get the files *„lcalc.he.5-0.000-1-3.26425130263651-1-c0-10000.txt“* and *„lcalc.co.5-0.000-1-3.26425130263651-1-c0-10000.txt“* which contains the header and the coefficients. By concatenating these files you get a valid input file for lcalc.

10 Plot

If we want to make a picture of a Maass waveform from the previous list we simply write

```
./maasswf -start 1 -stop 1  -f testlist.txt -plot
```

This produces the file *„graph5-0.000-1-6.96387424068007-200x200_-0.50-0.50x0.16-1.01.txt“* and copying this to *graph.txt* the following SAGE-code:

```
from pylab import *
X=MatrixSpace(RR,200)
X=load('path-to-file/graph.txt')
q=pcolor(X)
a=gca()
a.set_xticklabels(['-0.5','-0.25','0','0.25','0.5'])
a.set_yticklabels(['0.01','0.25','0.5','0.75','1.0'])
savefig('q.png')
```

can be used to produce a picture in png-format. The figure in this case is the left figure below. Since this eigenvalue is rather small ($R = 6.96\ldots$) the figure does not seem very interesting or „chaotic“ (the plotted function ϕ is real-valued, red corresponds to positive and blue to negative values). To demonstrate the large λ behavior, the right picture corresponds $R \approx 300$ (in this figure I plotted $|\phi|^2$ so the dark blue means that $|\phi|^2$ is close to zero).