Computing Bianchi Modular Forms

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THE GOAL

We want to compute Bianchi modular forms and the Hecke action on them.

Bianchi modular forms are modular forms for GL_2 over imaginary quadratic fields K. They can be seen as

$\star$ Certain real analytic functions on the hyperbolic 3-space $\mathcal{H}^3$,

$\star$ Regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_K)$,

$\star$ Certain classes in the cohomology of quotients of $\mathcal{H}^3$ by congruence subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$

We use the last approach.

THE ALGORITHM

The sheaf cohomology of the manifold $\Gamma \backslash \mathcal{H}^3$ is isomorphic to the group cohomology of its fundamental group Γ.

We compute the group cohomology $H^1(\mathrm{PSL}_2(\mathcal{O}_K), E_{k,l}(K))$.

• use finite presentations of $\mathrm{PSL}_2(\mathcal{O}_K)$.

e.g. $\mathrm{PSL}_2(\mathcal{O}_{-2}) = \langle a, b, u \mid b^2, (ab)^3, aua^{-1}u^{-1}, (bu^{-1}bu)^2 \rangle$

• identify cocyles with their images on the generators of $\mathrm{PSL}_2(\mathcal{O}_K)$ and find them inside the kernel of a matrix given by the relations of $\mathrm{PSL}_2(\mathcal{O}_K)$.

• compute the Hecke operators on cohomology using the Euclidean algorithm.

• use Shapiro's Lemma for $\Gamma < \mathrm{PSL}_2(\mathcal{O}_K)$.

THE IMPLEMENTATION

There are two prototype MAGMA functions presented on the author's webpage for $K = \mathbb{Q}(\sqrt{-2})$.

The function DIM(a,b,k,l) computes the dimensions of $H^1(\Gamma_0(a+b\sqrt{-2}), E_{k,l}(K))$, its cuspidal subspace and the dimensions of the GL_2-invariant subspaces of these two spaces.

The function HECKE(c,d,D) computes the characteristic polynomial and the eigenspaces for Hecke operator $T(c+d\sqrt{-2})$ on $H^1(\Gamma_0(a+b\sqrt{-2}), E_{k,l}(K))$.
The input D is a record that gives the vector space that is computed by the DIM function.

AN EXAMPLE

We take a look at $H^1(\Gamma_0(1+\sqrt{-2}), E_{3,3}(K))$.

```
>load "PATH1/bianchi.m";
> data:=DIM(1,1,3,3);
full,cusp [ 4, 2 ]
plus,cusp-plus [ 2, 0 ]
```

We see that the full space is 4 dimensional with a 2 dimensional cuspidal subspace. Moreover, the GL_2-invariant part of the cuspidal subspace is trivial. The relevant data is stored as a record that we named "data" which will be an input for the next function

AN EXAMPLE, cont'd

We compute the operator $T(1 - \sqrt{-2})$ on this space.

```
 > load "PATH2/hecke.m";
 > HECKE(1,-1,data);
[ <$.1 - 6, 2>, <$.1 + 14, 2> ]
[*  Echelonized basis:
(1 0 1/140481*(1940*w + 10240) 1/140481*(1552*w + 8192))
(0 1 1/5203*(-307*w + 96) 1/26015*(-1228*w + 384)), 6,
    Echelonized basis:
(1 0 0 0)
(0 0 0 1), -14
*]
```

We see from the factored characteristic polynomial that there are two eigenvalues 6 and -14. We see the vectors associated to these eigenvalues.